INTRODUCTION TO DIFFERENTIAL GALOIS THEORY

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Abstract

The course will involve only *complex analytic linear differential equations*. Thus, it will not be based on the general algebraic formalism of differential algebra, but rather on complex function theory.

There are two main approaches to Differential Galois Theory. The first one, usually called Picard-Vessiot Theory, and mainly developped by Kolchin, is in some sense a transposition of the Galois theory of algebraic equations in the form it was given by the german algebraists: to a (linear) differential equation, one attaches an extension of *differential* fields and one defines the Galois group of the equation as the group of automorphisms compatible with the differential structure. This group is automatically endowed with a structure of an algebraic group, and one must take in account that structure to get information on the differential equation. This approach has been extensively developped, it has given rise to computational tools (efficient algorithms and software) and it is well documented in a huge litterature.

A more recent approach is based on so-called "tannakian duality". It is very powerful and can be extended to situations where the Picard-Vessiot approach is not easily extended (like q-difference Galois theory). There is less litterature and it has a reputation of being very abstract. However, in some sense, the tannakian approach can be understood as an algebraic transposition of the Riemann-Hilbert correspondence. In this way, it is rooted in very concrete and down-to-earth processes: the analytic continuation of power series solutions obtained by the Cauchy theorem and the ambiguity introduced by the many-valuedness of the solutions. This is expressed by the *monodromy group*, a precursor of the differential Galois group, and by the *monodromy representation*. The Riemann-Hilbert correspondence is the other big galoisian theory of XIXth century, and it is likely that Picard had it in mind when he started to create Differential Galois Theory.

Therefore, I intend to devote the first part of the course to the study of the Riemann-Hilbert correspondence, which is, anyhow, a must for anyone who wants to work with complex differential equations. At the end of this first part, I will briefly sketch the way that Picard-Vessiot theory allows one to replace the monodromy group by an algebraic group. In the second part of the course, I shall introduce (almost from scratch) the basic tools required for using algebraic groups in Differential Galois Theory, whatever the approach (Picard-Vessiot or tannakian). In the third part of the course, I shall show how to attach algebraic groups and their representations to complex analytic linear differential equations.

Prerequisites. The main prerequisites are: complex function theory (mostly using power series); linear algebra (mostly reduction of matrices); elementary knowledge of groups and of polynomials in many variables. Each time a more advanced result will be needed, it will be precisely stated and explained and an easily accessible reference will be given.

Warning. The second and third part of the course introduce the students to more sophisticated techniques. The “fine tuning” of their contents will have to be adapted according to the reactions of the audience to the first part. Therefore, the present description may evolve at the moment of the teaching and typing of the corresponding chapters. This is particularly true of the third part.

Conventions. Notation $A := B$ means that the term A is defined by formula B. New terminology is written in *emphatic style* when first defined. Note that a definition can appear in the course of a theorem, an example, an exercice, etc.

Example 0.0.1 L'espace vectoriel $E^* := \mathrm{Hom}_K(E, K)$ est appelé *dual* de E.

A list of notations is provided at the end.

We mark the end of a proof, or its absence, by the symbol $\square$

Part I

A quick introduction to complex analytic functions

This “crash course” will include almost no proofs; I’ll give them only if they may serve as a training for the following parts.

The prerequisites for this part are: topology and analysis in $\mathbf{R}$ and in $\mathbf{R}^2$; complex numbers; linear algebra.

The reader may look for further information in:

- Ahlfors, “Complex analysis”;
- Cartan, “Elementary theory of analytic functions of one or several variables”.

The book “Real and complex analysis” can also be used.

Chapter 1

The complex exponential function

This is a *very important* function !

1.1 The series

For any $z \in \mathbf{C}$, we define:

$$\exp(z) := \sum_{n \geq 0} \frac{1}{n!} z^n = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots$$

On the closed disk

$$\overline{\mathrm{D}}(0,R) := \{z \in \mathbf{C} \mid |z| \leq R\},$$

one has $\left| \frac{1}{n!} z^n \right| \leq \frac{1}{n!} R^n$ and we know that the series $\sum\limits_{n \geq 0} \frac{1}{n!} R^n$ converges for any $R > 0$. Therefore, $\exp(z)$ is a normally convergent series of continuous functions, and $z \mapsto \exp(z)$ is *a continuous function from* $\mathbf{C}$ *to* $\mathbf{C}$.

Theorem 1.1.1 *For any* $a, b \in \mathbf{C}$, *one has* $\exp(a+b) = \exp(a)\exp(b)$.

Proof. - We just show the calculation, but this should be justified by arguments from real analysis (absolute convergence implies commutative convergence):

$$\begin{aligned}
\exp(a+b) &= \sum_{n \geq 0} \frac{1}{n!} (a+b)^n \\
&= \sum_{n \geq 0} \frac{1}{n!} \sum_{k+l=n} \frac{(k+l)!}{k!\,l!} a^k b^l \\
&= \sum_{\substack{n \geq 0 \\ k+l=n}} \frac{1}{n!} \frac{(k+l)!}{k!\,l!} a^k b^l \\
&= \sum_{k,l \geq 0} \frac{1}{k!\,l!} a^k b^l \\
&= \exp(a)\exp(b).
\end{aligned}$$

$\square$

We now give a list of basic, easily proved properties. First, the effect of complex conjugation:

$$\forall z \in \mathbf{C}\ ,\ \overline{\exp(z)} = \exp(\overline{z}).$$

Since obviously $\exp(0) = 1$, on draws from the theorem:

$$\forall z \in \mathbf{C}\ ,\ \exp(z) \in \mathbf{C}^* \text{ and} \exp(-z) = \frac{1}{\exp(z)}\cdot$$

Also, $z \in \mathbf{R} \Rightarrow \exp(z) \in \mathbf{R}^*$ and then, writing $\exp(z) = \big(\exp(z/2)\big)^2$, one sees that $\exp(z) \in \mathbf{R}_+^*$.

Last, if $z \in \mathrm{i}\mathbf{R}$ (pure imaginary), then $\overline{z} = -z$, so putting $w := \exp(z)$, one has $\overline{w} = w^{-1}$ so that $|w| = 1$. In other words, exp sends $\mathrm{i}\mathbf{R}$ to the unit circle $\mathbf{U} := \{z \in \mathbf{C} \mid |z| = 1\}$.

Summarizing, if $x := \mathrm{Re}(z)$ and $y := \mathrm{Im}(z)$, then exp sends z to $\exp(z) = \exp(x)\exp(\mathrm{i}y)$, where $\exp(x) \in \mathbf{R}_+^*$ and $\exp(\mathrm{i}y) \in \mathbf{U}$.

Exercice 1.1.2 For $z \in \mathbf{C}$, define $\cos(z) := \dfrac{\exp(z) + \exp(-z)}{2}$ and $\sin(z) := \dfrac{\exp(z) - \exp(-z)}{2\mathrm{i}}$, so that cos is an even function, sin is an odd function and $\exp(z) = \cos(z) + \mathrm{i}\sin(z)$. Translate the property of theorem 1.1.1 into properties of cos and sin.

1.2 The function exp is C-derivable

Lemma 1.2.1 *If $|z| \leq R$, then $|\exp(z) - 1 - z| \leq \dfrac{e^R}{2}|z|^2$.*

Proof. - $|\exp(z) - 1 - z| = \dfrac{z^2}{2}\left(1 + \dfrac{z}{3} + \dfrac{z^2}{12} + \cdots\right)$ and $\left|1 + \dfrac{z}{3} + \dfrac{z^2}{12} + \cdots\right| \leq 1 + \dfrac{R}{3} + \dfrac{R^2}{12} + \cdots \leq e^R$. $\square$

Theorem 1.2.2 *For any fixed $z_0 \in \mathbf{C}$:*

$$\lim_{h \to 0} \frac{\exp(z_0 + h) - \exp(z_0)}{h} = \exp(z_0).$$

Proof. - $\dfrac{\exp(z_0 + h) - \exp(z_0)}{h} = \exp(z_0)\dfrac{\exp(h) - 1}{h}$ and, after the lemma, $\dfrac{\exp(h) - 1}{h} \to 1$ when $h \to 0$. $\square$

Therefore, exp is derivable with respect to the complex variable: we say that it is **C**-derivable (we shall change terminology later) and that its **C**-derivative is itself, which we write $\dfrac{d\exp(z)}{dz} = \exp(z)$ or $\exp' = \exp$.

Corollary 1.2.3 *On $\mathbf{R}$, exp restricts to the usual real exponential function; that is, for $x \in \mathbf{R}$, $\exp(x) = e^x$.*

Proof. - The restricted function $\exp : \mathbf{R} \to \mathbf{R}$ sends 0 to 1 and it is its own derivative, so it is the usual real exponential function. □

For this reason, for now on, we shall put $e^z := \exp(z)$ when z is an arbitrary complex number.

Corollary 1.2.4 *For* $y \in \mathbf{R}$, *one has* $\exp(iy) = \cos(y) + i\sin(y)$.

Proof. - Put $f(y) := \exp(\mathrm{i}y)$ and $g(y) := \cos(y) + \mathrm{i}\sin(y)$. These functions satisfy $f(0) = g(0) = 1$ and $f' = \mathrm{i}f$, $g' = \mathrm{i}g$. Therefore the function $h := f/g$ which is well defined from $\mathbf{R}$ to $\mathbf{C}$ satisfies $h(0) = 1$ and $h' = 0$, so that it is constant equal to 1. □

Note that this implies the famous formula of Euler $e^{\mathrm{i}\pi} = -1$.

Corollary 1.2.5 *For* $x, y \in \mathbf{R}$, *one has* $e^{x+iy} = e^x(\cos y + i\sin y)$.

Corollary 1.2.6 *The exponential map* $\exp : \mathbf{C} \to \mathbf{C}^*$ *is surjective.*

Proof. - Any $w \in \mathbf{C}^*$ can be written $w = r(\cos\theta + \mathrm{i}\sin\theta)$, so $w = \exp(\ln(r) + \mathrm{i}\theta)$. □

The reader can find a proof which does not require previous knowledge of trigonometrical functions in the preliminary chapter of Rudin.

Exercice 1.2.7 Let $a, b > 0$ and $U := \{z \in \mathbf{C} \mid -a < \mathrm{Re}(z) < a \text{ and } -b < \mathrm{Im}(z) < b\}$ (thus, an open rectangle under the identification of $\mathbf{C}$ with $\mathbf{R}$). Assuming $b < \pi$, describe the image $V := \exp(U) \subset \mathbf{C}^*$ and define an inverse map $V \to U$.

The exponential viewed as a map $\mathbf{R}^2 \to \mathbf{R}^2$. It will be useful to consider functions $f : \mathbf{C} \to \mathbf{C}$ as functions $\mathbf{R}^2 \to \mathbf{R}^2$, under the usual identification of $\mathbf{C}$ with $\mathbf{R}^2$: $x + \mathrm{i}y \leftrightarrow (x, y)$. In this way, f is described by $(x,y) \mapsto F(x,y) := \big(A(x,y), B(x,y)\big)$, where $\begin{cases} A(x,y) := \mathrm{Re}\big(f(x+\mathrm{i}y)\big), \\ B(x,y) := \mathrm{Im}\big(f(x+\mathrm{i}y)\big). \end{cases}$

In the case where f is the exponential function exp, we compute easily:

$$\begin{cases} A(x,y) = e^x\cos(y), \\ B(x,y) = e^x\sin(y), \end{cases} \implies F(x,y) = \big(e^x\cos(y), e^x\sin(y)\big).$$

We are going to compare the differential of the map F with the $\mathbf{C}$-derivative of the exponential map. On the one hand, the differential $dF(x,y)$ is the linear map defined by the relation:

$$F(x+u, y+v) = F(x,y) + dF(x,y)(u,v) + o(u,v),$$

where $o(u,v)$ is small compared to the norm of (u,v) when $(u,v) \to (0,0)$. Actually, $dF(x,y)$ can be expressed using partial derivatives:

$$dF(x,y)(u,v) = \Big(\frac{\partial A(x,y)}{\partial x}u + \frac{\partial A(x,y)}{\partial y}v, \frac{\partial B(x,y)}{\partial x}u + \frac{\partial B(x,y)}{\partial y}v\Big).$$

Therefore, it is described by the Jacobian matrix:

$$JF(x,y) = \begin{pmatrix} \dfrac{\partial A(x,y)}{\partial x} & \dfrac{\partial A(x,y)}{\partial y} \\ \dfrac{\partial B(x,y)}{\partial x} & \dfrac{\partial B(x,y)}{\partial y} \end{pmatrix}$$

On the side of the complex function $f := \exp$, putting $z := x + \mathrm{i}y$ and $h := u + \mathrm{i}v$, we write:

$$f(z+h) = f(z) + hf'(z) + o(h), \text{ that is } \exp(z+h) = \exp(z) + h\exp(z) + o(h)$$

Here, the linear part is $f'(z)h = \exp(z)h$, so we draw the conclusion that (under our correspondance of $\mathbf{C}$ with $\mathbf{R}^2$):

$$hf'(z) \longleftrightarrow dF(x,y)(u,v),$$

that is, comparing real and imaginary parts:

$$\begin{cases} \dfrac{\partial A(x,y)}{\partial x}u + \dfrac{\partial A(x,y)}{\partial y}v = \mathrm{Re}(f'(z))u - \mathrm{Im}(f'(z))v, \\ \dfrac{\partial B(x,y)}{\partial x}u + \dfrac{\partial B(x,y)}{\partial y}v = \mathrm{Im}(f'(z))u + \mathrm{Re}(f'(z))v. \end{cases}$$

Since it must be true for all u, v, we conclude that:

$$JF(x,y) = \begin{pmatrix} \dfrac{\partial A(x,y)}{\partial x} & \dfrac{\partial A(x,y)}{\partial y} \\ \dfrac{\partial B(x,y)}{\partial x} & \dfrac{\partial B(x,y)}{\partial y} \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(f'(z)) & -\mathrm{Im}(f'(z)) \\ \mathrm{Im}(f'(z)) & \mathrm{Re}(f'(z)) \end{pmatrix}$$

As a consequence, the Jacobian determinant det $JF(x,y)$ is equal to $|f'(z)|^2$ and thus vanishes if and only if $f'(z) = 0$: in the case of the exponential function, it vanishes nowhere.

Exercice 1.2.8 Verify these formulas when $A(x,y) = e^x \cos(y)$, $B(x,y) = e^x \sin(y)$ and $f'(z) = \exp(x + \mathrm{i}y)$.

1.3 The exponential function as a covering map

From equation $e^{x+\mathrm{i}y} = e^x(\cos y + \mathrm{i}\sin y)$, one sees that $e^z = 1 \Leftrightarrow z \in 2\mathrm{i}\pi\mathbf{Z}$, *i.e.* $\exists k \in \mathbf{Z} \,:\, z = 2\mathrm{i}\pi k$. It follows that $e^{z_1} = e^{z_2} \Leftrightarrow e^{z_2 - z_1} = 1 \Leftrightarrow z_2 - z_1 \in 2\mathrm{i}\pi\mathbf{Z}$, *i.e.* $\exists k \in \mathbf{Z} \,:\, z_2 = z_1 + 2\mathrm{i}\pi k$. We shall write this relation: $z_2 \equiv z_1 \pmod{2\mathrm{i}\pi\mathbf{Z}}$ or more shortly $z_2 \equiv z_1 \pmod{2\mathrm{i}\pi}$.

Theorem 1.3.1 *The map* $\exp : \mathbf{C} \to \mathbf{C}^*$ *is a* covering map, *that is: for any* $w \in \mathbf{C}^*$, *there is a neighborhood* $V \subset \mathbf{C}^*$ *of* w *such that* $\exp^{-1}(V) = \bigsqcup U_k$ *(disjoint union), where each* $U_k \subset \mathbf{C}$ *is an open set and* $\exp : U_k \to V$ *is an homeomorphism (a bicontinuous bijection).*

Proof. - Choose a particular $z_0 \in \mathbf{C}$ such that $\exp(z_0) = w$. Choose an open neighborhood U_0 of z_0 such that, for any $z', z'' \in U_0$, one has $|z'' - z'| < 2\pi$. Then exp maps bijectively U_0 to $V := \exp(U_0)$. Moreover, one has $\exp^{-1}(V) = \bigsqcup U_k$ where k runs in $\mathbf{Z}$ and the $U_k = U_0 + 2\mathrm{i}\pi k$ are open sets. It remains to show that V is an open set. The most generalizable way is to use the local inversion theorem, since the Jacobian determinant vanishes nowhere. Another way is to choose an open set as in exercice 1.2.7. □

The fact that exp is a covering map is a very important topological property and it has many consequences.

Corollary 1.3.2 (Path lifting property) *Let $a < b$ in $\mathbf{R}$ and let $\gamma : [a,b] \to \mathbf{C}^*$ be a continuous path with origin $\gamma(a) = w_0 \in \mathbf{C}^*$. Let $z_0 \in \mathbf{C}$ be such that $\exp(z_0) = w_0$. Then, there exists a unique* lifting, *a continuous path $\overline{\gamma} : [a,b] \to \mathbf{C}^*$ such that $\forall t \in [a,b]$, $\exp \overline{\gamma}(t) = \gamma(t)$ and subject to the initial condition $\overline{\gamma}(a) = z_0$.*

Exercice 1.3.3 If one chooses another $z_0' \in \mathbf{C}$ such that $\exp(z_0') = w_0$, one gets another lifting $\overline{\gamma}' : [a,b] \to \mathbf{C}^*$ such that $\forall t \in [a,b]$, $\exp \overline{\gamma}'(t) = \gamma(t)$ and subject to the initial condition $\overline{\gamma}'(a) = z_0'$. Show that there is some *constant* $k \in \mathbf{Z}$ such that $\forall t \in [a,b]$, $\overline{\gamma}'(t) = \overline{\gamma}(t) + 2i\pi k$.

Corollary 1.3.4 (Index of a loop with respect to a point) *Let $\gamma : [a,b] \to \mathbf{C}^*$ be a continuous* loop, *that is $\gamma(a) = \gamma(b) = w_0 \in \mathbf{C}^*$. Then, for any lifting $\overline{\gamma}$ of γ, one has $\overline{\gamma}(b) - \overline{\gamma}(a) = 2i\pi n$ for some $n \in \mathbf{Z}$. The number n is the same for all the liftings, it depends only on the loop γ: it is the* index of γ around 0, *written $I(0,\gamma)$.*

Actually, another property of covering maps (the "homotopy lifting property) allows one to conclude that $I(0,\gamma)$ does not change if γ is continuously deformed within $\mathbf{C}^*$: it only depends on the "homotopy class" of γ (see the topology course).

Example 1.3.5 If $\gamma(t) = e^{nit}$ on $[0,2\pi]$, then all liftings of γ have the form $\overline{\gamma}(t) = nit + 2i\pi k$ for some $k \in \mathbf{Z}$ and one finds $I(0,\gamma) = n$.

1.4 The exponential of a matrix

For a complex vector $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbf{C}^n$, we define $||X||_\infty := \max\limits_{1 \leq i \leq n} (|x_i|)$. Then, for a complex square matrix $A = (a_{i,j})_{1 \leq i,j \leq n} \in \mathrm{Mat}_n(\mathbf{C})$, define the subordinated norm:

$$|||A|||_\infty := \sup_{\substack{X \in \mathbf{C}^n \\ X \neq 0}} \frac{||AX||_\infty}{||X||_\infty} = \max_{1 \leq i \leq n} \sum_{j=1}^{n} \left| a_{i,j} \right|.$$

Then, for the identity matrix, $|||I_n|||_\infty = 1$; and, for a product, $|||AB|||_\infty \leq |||A|||_\infty |||B|||_\infty$. It follows easily that $\left|\left|\left|\frac{1}{k!}A^k\right|\right|\right|_\infty \leq \frac{1}{k!} |||A|||_\infty^k$ for all $k \in \mathbf{N}$, so that the series $\sum\limits_{k \geq 0} \frac{1}{k!} A^k$ converges absolutely for any $A \in \mathrm{Mat}_n(\mathbf{C})$. It actually converges normally on all compacts and therefore define a continuous map $\exp : \mathrm{Mat}_n(\mathbf{C}) \to \mathrm{Mat}_n(\mathbf{C})$, $A \mapsto \sum\limits_{k \geq 0} \frac{1}{k!} A^k$. We shall also write for short $e^A := \exp(A)$. In the case $n = 1$, the notation is consistent.

Examples 1.4.1 (i) For a diagonal matrix $A := \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$, one has $\frac{1}{k!} A^k = \mathrm{Diag}(\lambda_1^k/k!, \ldots, \lambda_n^k/k!)$, so that $\exp(A) = \mathrm{Diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})$.

(ii) If A is an upper triangular matrix with diagonal $D := \mathrm{Diag}(\lambda_1, \dots, \lambda_n)$, then $\dfrac{1}{k!}A^k$ is an upper triangular matrix with diagonal $\dfrac{1}{k!}D^k$, so that $\exp(A)$ is an upper triangular matrix with diagonal $\exp(D) = \mathrm{Diag}(e^{\lambda_1}, \dots, e^{\lambda_n})$. Similar relations hold for lower triangular matrices.
(iii) Take $A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $A^2 = I_2$, so that $\exp(A) = aI_2 + bA = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $a = \sum_{k \geq 0} \dfrac{1}{(2k)!}$ and $b = \sum_{k \geq 0} \dfrac{1}{(2k+1)!}$.

The same kind of calculations as for the exponential map gives the rules $\exp(0_n) = I_n$; $\exp(\overline{A}) = \overline{\exp(A)}$; and:

$$AB = BA \Longrightarrow \exp(A+B) = \exp(A)\exp(B) = \exp(B)\exp(A).$$

Remark 1.4.2 The condition $AB = BA$ is required to use the Newton binomial formula. If we take for instance $A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then $AB \neq BA$. We have $A^2 = B^2 = 0$, so that $\exp(A) = I_2 + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\exp(B) = I_2 + B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, thus $\exp(A)\exp(B) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. On the other hand, $A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the previous example gave the value of $\exp(A+B)$, which was clearly different.

It follows from the previous rules that $\exp(-A) = \big(\exp(A)\big)^{-1}$ so that exp actually sends $\mathrm{Mat}_n(\mathbf{C})$ to $\mathrm{GL}_n(\mathbf{C})$. Now there are rules more specific to matrices. For the transpose, using the fact that ${}^t(A^k) = ({}^tA)^k$, and also the continuity of $A \mapsto {}^tA$ (this is required to go to the limit in the infinite sum), we see that $\exp({}^tA) = {}^t(\exp(A))$. Last, if $P \in \mathrm{GL}_n(\mathbf{C})$, from the relation $(PAP^{-1})^n = PA^nP^{-1}$ (and also from the continuity of $A \mapsto PA^nP^{-1}$), we deduce the very useful equality:

$$P\exp(A)P^{-1} = \exp(PAP^{-1}).$$

Now any complex matrix A is conhugate to an upper triangular matrix T having the eigenvalues of A on the diagonal; using the examples above, one concludes that if A has eigenvalues $\lambda_1, \dots, \lambda_n$, then $\exp(A)$ has eigenvalues $e^{\lambda_1}, \dots, e^{\lambda_n}$:

$$\mathrm{Sp}(e^A) = e^{\mathrm{Sp}(A)}.$$

Note that this implies $\mathrm{Tr}(e^A) = e^{\det A}$.

Example 1.4.3 Let $A := \begin{pmatrix} 0 & -\pi \\ \pi & 0 \end{pmatrix}$. Then A is diagonalisable with spectrum $\mathrm{Sp}(A) = \{\mathrm{i}\pi, -\mathrm{i}\pi\}$. Thus, $\exp(A)$ is diagonalisable with spectrum $\{-1, -1\}$. Therefore, $\exp(A) = -I_2$.

Exercice 1.4.4 Compute $\exp \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$ in two ways: by diagonalisation as in the example above; by direct calculation as in a previous example. Deduce from this the value of $\exp \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

1.5 Application to differential equations

Let $A \in \mathrm{Mat}_n(\mathbf{C})$ be fixed. Then, $z \mapsto e^{zA}$ is a $\mathbf{C}$-derivable function from $\mathbf{C}$ to the complex linear space $\mathrm{Mat}_n(\mathbf{C})$; this simply means that each coefficient is a $\mathbf{C}$-derivable function from $\mathbf{C}$ to itself. Derivating our matrix-valued function coefficientwise, we find:

$$\frac{d}{dz}e^{zA} = Ae^{zA} = e^{zA}A.$$

Indeed, $\dfrac{e^{(z+h)A} - e^{zA}}{h} = e^{zA}\dfrac{e^{hA} - I_n}{h} = \dfrac{e^{hA} - I_n}{h}e^{zA}$ and $\dfrac{e^{hA} - I_n}{h} = A + \dfrac{h}{2}A^2 + \cdots$

Now consider the vectorial differential equation:

$$\frac{d}{dz}X(z) = AX(z),$$

where $X : \mathbf{C} \to \mathbf{C}^n$ is searched as a $\mathbf{C}$-derivable vector-valued function, and again derivation is performed coefficientwise. We solve this by changing of unknown function: $X(z) = e^{zA}Y(z)$. Then, applying Leibniz rule for derivation: $(fg)' = f'g + fg'$ (it works the same for $\mathbf{C}$-derivation), we find:

$$X' = AX \Longrightarrow e^{zA}Y' + Ae^{zA}Y = Ae^{zA}Y \Longrightarrow e^{zA}Y' = 0 \Longrightarrow Y' = 0.$$

Therefore, $Y(z)$ is a constant function. (Again, we admit a property of $\mathbf{C}$-derivation: that $f' = 0 \Rightarrow f$ constant.) If now we fix $z_0 \in \mathbf{C}$, $X_0 \in \mathbf{C}^n$ and we adress the Cauchy problem:

$$\begin{cases} \dfrac{d}{dz}X(z) = AX(z), \\ X(z_0) = X_0, \end{cases}$$

we see that the unique solution is $X(z) := e^{(z-z_0)A}X_0$.

An important theoretical consequence is the following. Call $Sol(A)$ the set of solutions of $\dfrac{d}{dz}X(z) = AX(z)$. This is obviously a complex linear space. What we proved is that the map $X \mapsto X(z_0)$ from $Sol(A)$ to $\mathbf{C}^n$, which is obviously linear, is also bijective. Therefore, it is an isomorphism of $Sol(A)$ with $\mathbf{C}^n$. (This is a very particular case of the Cauchy theorem for *complex* differential equations.)

Example 1.5.1 To solve the linear homogeneous second order scalar equation (with constant coefficients) $f'' + pf' + qf = 0$ ($p, q \in \mathbf{C}$), we introduce the vector valued function $X(z) := \begin{pmatrix} f(z) \\ f'(z) \end{pmatrix}$ and find that our scalar equation is actually equivalent to the vector equation:

$$X' = AX, \text{ where } A := \begin{pmatrix} 0 & 1 \\ -q & p \end{pmatrix}.$$

Therefore, the solution will be searched in the form $X(z) := e^{(z-z_0)A}X_0$, where z_0 may be chosen at will or else imposed by initial conditions.

Exercice 1.5.2 Compute $e^{(z-z_0)A}$ and solve the problem with initial conditions $f(0) = a$, $f'(0) = b$. There will be a discussion according to whether $p^2 - 4q = 0$ or $\neq 0$.

Chapter 2

Power series

2.1 Formal power series

These are actually purely algebraic objects, a kind of "polynomials of infinite degree":

$$\mathbf{C}[[z]] := \{\sum_{n\geq 0} a_n z^n \mid \forall n \in \mathbf{N}\,,\ a_n \in \mathbf{C}\}.$$

This means that we do not attach (for the moment) any meaning to the "sum", do not consider z as a number and do not see f as a function. We agree to say that the formal power series $\sum\limits_{n\geq 0} a_n z^n$ and $\sum\limits_{n\geq 0} b_n z^n$ are equal if, and only if, they have the same coefficients.

Let $\lambda \in \mathbf{C}$ and let $\sum\limits_{n\geq 0} a_n z^n \in \mathbf{C}[[z]]$ and $\sum\limits_{n\geq 0} b_n z^n \in \mathbf{C}[[z]]$. Then we define the following operations:

$$\begin{aligned}
\lambda.\left(\sum_{n\geq 0} a_n z^n\right) &:= \sum_{n\geq 0} (\lambda.a_n) z^n, \\
\sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} b_n z^n &:= \sum_{n\geq 0} (a_n + b_n) z^n, \\
\left(\sum_{n\geq 0} a_n z^n\right).\left(\sum_{n\geq 0} b_n z^n\right) &:= \sum_{n\geq 0} c_n z^n \text{ where } \forall n \in \mathbf{N}\,,\ c_n := \sum_{i+j=n} a_i b_j.
\end{aligned}$$

With the first and second operation, we make $\mathbf{C}[[z]]$ into a linear space over the complex numbers. With the second and last operation, we make it into a commutative ring. (Its zero element is $\sum\limits_{n\geq 0} 0z^n$, written for short 0; its unit element is $1 + \sum\limits_{n\geq 1} 0.z^n$, written for short 1.) Altogether, we say that $\mathbf{C}[[z]]$ is a $\mathbf{C}$-algebra.

Polynomials can be considered as formal power series, with almost all their coefficients being zero. The operations are the same, so we identify $\mathbf{C}[z] \subset \mathbf{C}[[z]]$ as a sub-algebra (sub-linear space and sub-ring). Among polynomials are the constants: $\mathbf{C} \subset \mathbf{C}[z] \subset \mathbf{C}[[z]]$ and so we identify $a \in \mathbf{C}$ with $a + \sum\limits_{n\geq 1} 0.z^n \in \mathbf{C}[[z]]$.

Remark 2.1.1 Although it has no meaning for the moment to substitute a complex number $z_0 \in \mathbf{C}$ to the formal indeterminate z, we will allow ourselves to write $f(0) := a_0$ when $f := \sum_{n \geq 0} a_n z^n \in \mathbf{C}[[z]]$ and call it the constant term. It has the natural properties that $(\lambda.f)(0) = \lambda.f(0)$, $(f+g)(0) = f(0)+g(0)$ and $(f.g)(0) = f(0)g(0)$.

Invertible elements. In $\mathbf{C}[z]$, only nonzero constants are invertible, but in $\mathbf{C}[[z]]$ we can perform such calculations as $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$ Remember that we do not attach any numerical meaning to this equality, it only means that performing the product $(1+z)(1-z+z^2-z^3+\cdots)$ according to the rules above yields the result $1 + \sum_{n \geq 1} 0.z^n = 1$. More generally:

$$\left(\sum_{n \geq 0} a_n z^n\right) \cdot \left(\sum_{n \geq 0} b_n z^n\right) = 1 \Longleftrightarrow a_0 b_0 = 1 \text{ and } \forall n \geq 1 \,,\ \sum_{i+j=n} a_i b_j = 0$$
$$\Longleftrightarrow b_0 = 1/a_0 \text{ and } \forall n \geq 1 \,,\ b_n = -(a_n b_0 + \cdots + a_0 b_{n-1})/a_0.$$

Therefore, $\sum_{n \geq 0} a_n z^n$ is invertible if, and only if $a_0 \neq 0$. The coefficients of it inverse are then calculated by the above recursive formulas. The *group of units (invertible elements)* of the ring $\mathbf{C}[[z]]$ is:

$$\mathbf{C}[[z]]^* = \{\sum_{n \geq 0} a_n z^n \in \mathbf{C}[[z]] \mid a_0 \neq 0\} = \{f \in \mathbf{C}[[z]] \mid f(0) \neq 0\}.$$

Valuation. For $f := \sum_{n \geq 0} a_n z^n \in \mathbf{C}[[z]]$, we define $v_0(f) := \min\{n \in \mathbf{N} \mid a_n \neq 0\}$. This is called the *valuation of f* or the *order of f at* 0. By convention, $v_0(0) := +\infty$. Thus, if $v_0(f) = k \in \mathbf{N}$, then $f = a_k z^k + a_{k+1} z^{k+1} + \cdots$ and $a_k \neq 0$, and therefore $f = z^k u$ where $u \in \mathbf{C}[[z]]^*$. It easily follows that $f|g$, meaning "f divides g" (*i.e.* $\exists h \in \mathbf{C}[[z]] \;:\; g = fh$) if and only if $v_0(f) \leq v_0(g)$. Likewise, $f \in \mathbf{C}[[z]]^* \Leftrightarrow v_0(f) = 0$. Other useful rules are: $v_0(f+g) \geq \min(v_0(f), v_0(g))$ and $v_0(fg) = v_0(f) + v_0(g)$. An easy consequence is that $\mathbf{C}[[z]]$ is an integral ring, *i.e.* $fg = 0 \Rightarrow f = 0$ or $g = 0$.

Field of fractions. Let $f, g \neq 0$. If $f|g$, then $g/f \in \mathbf{C}[[z]]$. Otherwise, writing $f = z^k u$ and $g = z^l v$ with $k, l \in \mathbf{N}$ and $u, v \in \mathbf{C}[[z]]^*$ (thus $k = v_0(f)$ and $l = v_0(g)$), we have $g/f = z^{l-k}(v/u)$, where $l - k < 0$ (since f does not divide g) and $v/u \in \mathbf{C}[[z]]^*$. For example:

$$1/(z+z^2) = (1 - z + z^2 - z^3 + \cdots)/z = z^{-1} - 1 + z - z^2 + \cdots$$

This means that quotients of formal power series are "extended" formal power series with a finite number of negative powers. We therefore define the set of *formal Laurent series*:

$$\mathbf{C}((z)) := \mathbf{C}[[z]][z^{-1}] = \mathbf{C}[[z]] + z^{-1}\mathbf{C}[z^{-1}].$$

With the same operations as in $\mathbf{C}[[z]]$, it actually becomes a field: indeed, it is the field of fractions of $\mathbf{C}[[z]]$. Beware that the elements of $\mathbf{C}((z))$ have the form $\sum_{n >> -\infty,} a_n z^n$, the symbol $n >> -\infty$ meaning "$n \geq n_0$ for some $n_0 \in \mathbf{Z}$".

Formal derivation. For $f := \sum_{n\geq 0} a_n z^n \in \mathbf{C}[[z]]$, we define its derivative $f' = \sum_{n\geq 0} n a_n z^{n-1} = \sum_{n\geq 0} (n+1)a_{n+1}z^n$. We also write it df/dz. For the moment, this has no analytical meaning, it is just an algebraic operation on the coefficients. However, it satisfies the usual rules: if $\lambda \in \mathbf{C}$ then $(\lambda f)' = \lambda f'$; for any f,g, $(f+g)' = f' + g'$ and $(fg)' = fg' + f'g$ (Leibniz rule). Last, $f' = 0 \Leftrightarrow f \in \mathbf{C}$. Actually, the definition above as well as the rules can be extended without problem to formal Laurent series $f \in \mathbf{C}((z))$, and we have two more rules: $(1/f)' = -f'/f^2$ and $(f/g)' = (f'g - fg')/g^2$. If we introduce the *logarithmic derivative* f'/f, we conclude that $(fg)'/(fg) = f'/f + g'/g$ and that $(f/g)'/(f/g) = f'/f - g'/g$.

Example 2.1.2 (Newton binomial formula) These rules sometimes allow one to transform an algeraic equation into a differential one, which may be easier to deal with, as we shall see. Let $p \in \mathbf{Z}$ and $q \in \mathbf{N}$, and assume they are coprime and $q > 1$. Therefore, $r := p/q \in \mathbf{Q} \setminus \mathbf{Z}$ is a rational in reduced form and not an integer. We are going to define $f := (1+z)^r$ by requiring that $f = a_0 + a_1 z + \cdots$ has constant term $f(0) = a_0 = 1$ and that $f^q = (1+z)^p$. Now, using the fact that $h = 1$ is equivalent to $h'/h = 0$ and $h(0) = 1$, one has the logical equivalences:

$$\begin{aligned} f^q = (1+z)^p &\Longleftrightarrow f^q/(1+z)^p = 1 \\ &\Longleftrightarrow qf'/f = p/(1+z) \text{ and } a_0^q = 1 \\ &\Longleftrightarrow (1+z)f' = rf, \end{aligned}$$

since we have already required that $a_0 = 1$. Thus, we have a kind of Cauchy problem: a differential equation and an initial condition. Now, $(1+z)f' = \sum_{n\geq 0} ((n+1)a_{n+1} + na_n)z^n$, so, by identification of coefficients, we see that our Cauchy problem is equivalent to:

$$\begin{aligned} a_0 = 1 \text{ and } \forall n \geq 0\,,\ (n+1)a_{n+1} + na_n = ra_n &\Longleftrightarrow a_0 = 1 \text{ and } \forall n \geq 0\,,\ (n+1)a_{n+1} = (r-n)a_n \\ &\Longleftrightarrow a_0 = 1 \text{ and } \forall n \geq 0\,,\ a_{n+1} = \frac{r-n}{n+1}a_n \\ &\Longleftrightarrow \forall n \geq 0\,,\ a_n = \binom{r}{n}, \end{aligned}$$

where we have defined the *generalized binomial coefficients*:

$$\binom{r}{0} := 1 \text{ and } \forall n \geq 1\,,\ \binom{r}{n} := \frac{r(r-1)\cdots(r-n+1)}{n!}.$$

This gives the *generalized Newton binomial formula*:

$$(1+z)^r = \sum_{n\geq 0} \binom{r}{n} z^n.$$

Note that the right hand side makes sense for any $r \in \mathbf{C}$. After the study of the series $\log(1+z)$, we shall be able to see that then the left hand side is $\exp(r\log(1+z))$.

Substitution (or composition). Let $f,g \in \mathbf{C}[[z]]$. If they were functions, we could define the composition $g \circ f$. We shall define a formal analogue under some conditions, but rather call it *substitution of z by f in g*. The restrictive assumption is that $k := v_0(f) \geq 1$, *i.e.* $f(0) = 0$. On the other hand, we can authorize g to be a formal Laurent series. If we write $g = \sum_{n \geq n_0} b_n z^n$, we consider first its truncated series: $g_N := \sum_{n=n_0}^{N} b_n z^n$. These are "Laurent polynomials", that is polynomials with a some negative exponents. There are only a finite number of terms, so that it makes sense to define:

$$g_N \circ f := \sum_{n=n_0}^{N} b_n f^n.$$

Now, two successive composites $g_N \circ f$ and $g_{N-1} \circ f$ differ by the term $b_N f^N$, which has order Nk (or vanishes if $b_N = 0$). Therefore, the terms up to degree Nk are the same for all $g_{N'} \circ f$ for $N' \geq N$. In this way, we see that the coefficients "stabilize" and we can define $g \circ f$ as the limit coefficientwise of the $g_N \circ f$. The composition of formal power series satisfies the usual rules for functions. For instance, $(g \circ f)(0) = g(0)$ (this only makes sense if $g \in \mathbf{C}[[z]]$), $(g \circ f_1) \circ f_2 = g \circ (f_1 \circ f_2)$, $(g_1 + g_2) \circ f = (g_1 \circ f) + (g_2 \circ f)$, $(g_1 g_2) \circ f = (g_1 \circ f)(g_2 \circ f)$ and $(g \circ f)' = (g' \circ f) f'$.

Examples 2.1.3 (i) If $g = z + z^2$, writing $f = a_1 z + a_2 z^2 + \cdots$:

$$\begin{aligned}
g \circ f &= f + f^2 \\
&= \sum_{n \geq 1} \left(a_n + \sum_{i+j=n} a_i a_j \right) z^n \text{ and} \\
f \circ g &= f(z + z^2) \\
&= \sum_{n \geq 1} a_n (z + z^2)^n \\
&= \sum_{n \geq 1} \left(\sum_{i+j=n} \binom{i}{j} a_i \right) z^n.
\end{aligned}$$

(ii) Take $g = 1 + z + z^2/2 + z^3/6 + \cdots$ (the formal series with the same coefficients as exp) and $f(z) = z - z^2/2 + z^3/3 - z^4/4 + \cdots$ (this one corresponds in some sense to $\log(1+z)$). Then the very serious reader could calculate the first terms of $g \circ f$ and find $1 + z$ and then all terms seem to vanish. This can also be seen in the following way: clearly, $g' = g$ and $f' = 1/(1+z)$, so that putting $h := g \circ f$, one has $h' = h/(1+z)$ and $h(0) = 1$, from which one deduces easily that $h = 1 + z$.

Reciprocation. It is obvious that $g \circ z = g$ and that, when defined, $z \circ f = f$. So z is a kind of neutral element for the (non commutative) law $\circ$. One can therefore look for an inverse[1]: g being given, does there exist f such that $g \circ f = f \circ g = z$? For this to make sense, one must require that $v_0(f), v_0(g) \geq 1$. Then one sees easily that $v_0(g \circ f) = v_0(g) v_0(f)$, so that g can be reciprocated only if $v_0(g) = 1$. This condition is sufficient, and the solution of any of the two problems $g \circ f = z$ or $f \circ g = z$ is unique, and it is a solution of the other problem; but this is rather complicated to prove (see the book of Cartan).

Exercice 2.1.4 Solve the two problems when $g = z + z^2$.

[1]To distinguish this process for plain inversion $1/f$, we shall call it *reciprocation*.

2.2 Convergent power series

Theorem 2.2.1 *Suppose that the series* $f := \sum_{n \geq 0} a_n z_0^n$ *converges in* $\mathbf{C}$ *for some non zero value* $z_0 \in \mathbf{C}$. *Let* $R := |z_0|$. *Then the series* $\sum_{n \geq 0} a_n z^n$ *is normally convergent in any closed disk* $\overline{D}(0, R')$ *with* $0 < R' < R$.

Proof. - The $\left|a_n z_0^n\right| = |a_n| R^n$ are bounded by some $M > 0$ and, on $\overline{\mathrm{D}}(0, R')$, one has $|a_n z^n| \leq M(R'/R)^n$. □

Corollary 2.2.2 *The map* $z \mapsto \sum_{n \geq 0} a_n z^n$ *defines a continuous function on the open disk* $\overset{\circ}{\mathrm{D}}(0, R)$, *with values in* $\mathbf{C}$.

Definition 2.2.3 The *radius of convergence* (improperly abreviated as "r.o.c.") of the series $\sum_{n \geq 0} a_n z^n$ is defined as $\sup\{|z_0| \mid \sum_{n \geq 0} a_n z_0^n \text{ converges}\}$. If the radius of convergence of $f := \sum_{n \geq 0} a_n z^n$ is strictly positive, we call f a *power series*. If necessary, we emphasize: convergent power series. The set of power series is written $\mathbf{C}\{z\}$. (It is a subset of $\mathbf{C}[[z]]$ and it contains $\mathbf{C}[z]$.)

Examples 2.2.4 The r.o.c. of $\sum z^n$ is 1. The r.o.c. of $\sum z^n/n!$ is $+\infty$. The r.o.c. of $\sum n! z^n$ is 0.

Corollary 2.2.5 *Let* r *be the r.o.c. of* $\sum_{n \geq 0} a_n z^n$. *If* $|z_0| < r$, *then* $\sum_{n \geq 0} a_n z_0^n$ *is absolutely convergent. If* $|z_0| > r$, *then* $\sum_{n \geq 0} a_n z_0^n$ *diverges.*

The open disk $\overset{\circ}{\mathrm{D}}(0, r)$ is called the *disk of convergence*. Its boundary, the circle $\partial\mathrm{D}(0, r)$, is called the *circle of indeterminacy*.

Examples 2.2.6 Let $k \in \mathbf{Z}$ (actually, what follows works for $k \in \mathbf{R}$). The series $\sum_{n \geq 1} z^n/n^k$ converges absolutely for $|z| < 1$ and it diverges for $|z| > 1$, so its r.o.c. is 1.
For $k \leq 0$, it converges at no point of the circle of indeterminacy.
For $k > 1$, it converges at all points of the circle of indeterminacy.
For $0 < k \leq 1$, it diverges at $z = 1$. But it converges at all the other points of the circle of indeterminacy.

Exercice 2.2.7 Prove the last statement. (This uses Abel's transform: putting $S_n := 1 + z + \cdots + z^n$, one has

$$\sum_{n=1}^{N} z^n/n^k = \sum_{n=1}^{N} (S_n - S_{n-1})/n^k = \sum_{n=1}^{N} S_n(1/n^k - 1/(n+1)^k) + R_N,$$

where $R_N \to 0$ and, if $|z| = 1$, $z \neq 1$, the S_n are bounded.)

Rules for computing the radius of convergence. The first rule is the easiest to use; it is a direct consequence of d'Alembert criterion for series. If $a_n \neq 0$ for n big enough and if $\lim\limits_{n\to+\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = l$, then the r.o.c. is $1/l$.
The second rule is due to Hadamard. it is more complicated, bur more general: it always applies if the first rule applies, but it also applies in other cases. In its simplest form, it says that if $\lim\limits_{n\to+\infty} \left|\sqrt[n]{a_n}\right| = l$, then the r.o.c. is $1/l$. (There is a more complete form using $\overline{\lim}$ but we shall not need it.)

What happens near the circle of indeterminacy. The power series $f := \sum\limits_{n\geq 0} a_n z^n$ with r.o.c. r defines a continuous function on $\overset{\circ}{\mathrm{D}}(0,r)$. We shall admit:

Theorem 2.2.8 (Abel's radial theorem) *Suppose that f converges at some $z_0 \in \partial D(0,r)$. Then:*

$$f(z_0) = \lim_{\substack{t\to 1\\ t<1}} f(tz_0).$$

□

Example 2.2.9 Take $f(z) := \sum\limits_{n\geq 1} \dfrac{(-1)^{n-1}}{n} z^n$. Its r.o.c. is 1. By the standard criterion for alternating series, it converges at $z = 1$. For $0 < t < 1$, f can be derived termwise to give $\dfrac{1}{1+t}$, so that $f(t) = \ln(1+t)$. Therefore $\sum\limits_{n\geq 1} \dfrac{(-1)^{n-1}}{n} z^n = \ln 2$.

Remark 2.2.10 The converse of Abel's theorem is not generally true. For instance, if we take $f(z) := \sum\limits_{n\geq 0} (-1)^n z^n$, we see that for $0 < t < 1$, one has $f(t) = \dfrac{1}{1+t}$ which tends to $1/2$ as $t \to 1$, $t < 1$. But of course $f(1)$ does not converge. (When a converse is proved to be true under additional assumption, it is called a "Tauberian theorem".)

2.3 The ring of power series

To each power series f with strictly positive r.o.c., we associate a continuous function on some open neighborhood of 0 (actually a disk), which we also write f. The neighborhood is not the same for all power series and all the associated functions.

Lemma 2.3.1 *If two power series define the same function in some neighborhood of* 0*, then they are equal,* i.e. *they have the same coefficients.*

Proof. - Suppose $\sum\limits_{n\geq 0} a_n z^n = \sum\limits_{n\geq 0} b_n z^n$ for all z such that $|z| < r$, for some $r > 0$. Then putting $z = 0$ we have $a_0 = b_0$; then dividing by z, we see that $\sum\limits_{n\geq 0} a_{n+1} z^n = \sum\limits_{n\geq 0} b_{n+1} z^n$ for all z such that $0 < |z| < r$, hence also for $z = 0$ by continuity. Therefore, we can iterate the process. □

Therefore, in order to determine the power series f, it is enough to know the function f in some undetermined neighborhood of 0. We shall say that two functions defined in some neihborhoods of 0 (maybe not the same neighborhood) define the same *germ* at 0 if they are equal in some neighborhood of 0 (maybe strictly smaller than the intersection of the neighborhoods we begun with). So, to each power series is associated a germ at 0 and the process is injective. The set of germs obtained in this way (that is, coming from convergent power series) will be written $\mathcal{O}_0$. Therefore, we can identify the power series f with the associated germ and the set $\mathbf{C}\{z\}$ with the set $\mathcal{O}_0$.

Let us write temporarily $f_1 \sim f_2$ if f_1, f_2 are functions in some neighborhoods of 0 and if they define the same germ in $\mathcal{O}_0$. Then the following rules are easily established (with obvious notations): $f_1 \sim f_2 \Rightarrow \lambda f_1 \sim \lambda f_2$; $f_1 \sim f_2$ and $g_1 \sim g_2 \Rightarrow f_1 + g_1 \sim f_2 + g_2$; $f_1 \sim f_2$ and $g_1 \sim g_2 \Rightarrow f_1 g_1 \sim f_2 g_2$. We deduce from these rules that germs can be multiplied by scalars, and added and multiplied among themselves. Clearly, they form a $\mathbf{C}$-algebra.

On the other hand, it is not difficult to see that for power series, if f has r.o.c. r and defines the germ ϕ, then λf has r.o.c. r (or maybe $+\infty$ if $\lambda = 0$) and it defines the germ $\lambda\phi$. Likewise, if f and g respectively have r.o.c. r and s and define the germs ϕ and γ, then $f + g$ has r.o.c. $\geq r$ and defines the germ $\phi + \gamma$; and fg has r.o.c. $\geq r$ and defines the germ $\phi\gamma$. If f has r.o.c. r and $f(0) \neq 0$, then its inverse series in $\mathbf{C}[[z]]$ is a convergent power series[2].

We conclude that $\mathbf{C}\{z\}$ is a subalgebra of $\mathbf{C}[[z]]$ and that it is isomorphic to $\mathcal{O}_0$. Its invertible elements are those such that $f(0) \neq 0$.

Now let $f, g \in \mathbf{C}\{z\}$. If $f(0) = 0$, then one can compose the series. One can prove that $g \circ f$ is a convergent power series (see the book of Cartan for precisions) and that the associated germ is the composition of the germs associated to f and g. In the same way, the reciprocation processes for power series and for functions and germ correspond to each other.

Exercice 2.3.2 Taking $g = 1 + z + z^2/2 + z^3/6 + \cdots$ and $f(z) = z - z^2/2 + z^3/3 - z^4/4 + \cdots$, use the relation $g \circ f = 1 + z$ (proved in the section on formal power series) to compute $\sum_{n \geq 1} \mathrm{i}^n/n$. Then deduce the formulas: $1 - 1/3 + 1/5 - 1/7 + \cdots = \pi/4$ and $1/2 - 1/4 + 1/6 - 1/8 + \cdots = (\ln 2)/2$.

[2] It will follow from the next chapter on analytic functions that the r.o.c. of $1/f$ is the smallest $|z_0|$ for $f(z_0) = 0$; or is at least that of f if f has no zero in its disk of convergence.

2.4 C-derivability of power series

First, when h is small, $(z+h)^n = z^n + nz^{n-1}h + O(h^2)$, so that $\lim_{h\to 0} \dfrac{(z+h)^n - z^n}{h} = nz^{n-1}$. For a power series $f := \sum_{n\geq 0} a_n z^n$, we can therefore calculate formally the $\mathbf{C}$-derivative:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0} \sum_{n\geq 0} a_n \frac{(z+h)^n - z^n}{h} \\
&= \sum_{n\geq 0} \lim_{h\to 0} a_n \frac{(z+h)^n - z^n}{h} \\
&= \sum_{n\geq 0} a_n (nz^{n-1}) \\
&= \sum_{n\geq 0} (n+1) a_{n+1} z^n.
\end{aligned}$$

The interchange $\lim_{h\to 0} \sum_{n\geq 0} = \sum_{n\geq 0} \lim_{h\to 0}$ can be justified on the disk of convergence by the fact that the result converges normally in every strictly smaller closed disk. We conclude that convergent power series are $\mathbf{C}$-derivable and that the $\mathbf{C}$-derivation is computed in the same way as the formal derivation. Note that one cannot conclude on the circle of indeterminacy, as shows the example of the series $\sum_{n\geq 1} z^n/n^2$.

Theorem 2.4.1 *A power series of r.o.c. r defines on $\overset{\circ}{D}(0,r)$ an indefinitely $\mathbf{C}$-derivable function which is equal to its Taylor expansion at* 0.

Proof. - By iterating the argument above, one finds that the k^{th} derivative is $f^{(k)}(z) = \sum_{n\geq 0} \dfrac{(n+k)!}{n!} a_{n+k} z^n$, whence $a_k = \dfrac{f^{(k)}(0)}{k!} \cdot$ □

By exactly the same computation as in the case of the exponential, we draw:

Corollary 2.4.2 *The associated function $F(x,y) = (A(x,y), B(x,y))$ from $\overset{\circ}{D}(0,r)$ (viewed as an open disk in $\mathbf{R}^2$) to $\mathbf{R}^2$ is indefinitely differentiable. Its Jacobian matrix is given by the formula:*

$$JF(x,y) = \begin{pmatrix} \dfrac{\partial A(x,y)}{\partial x} & \dfrac{\partial A(x,y)}{\partial y} \\ \dfrac{\partial B(x,y)}{\partial x} & \dfrac{\partial B(x,y)}{\partial y} \end{pmatrix} = \begin{pmatrix} Re(f'(z)) & -Im(f'(z)) \\ Im(f'(z)) & Re(f'(z)) \end{pmatrix}$$

In particular, we have the Cauchy-Riemann formulas:

$$\begin{aligned}
\frac{\partial A(x,y)}{\partial x} &= \frac{\partial B(x,y)}{\partial y}, \\
\frac{\partial A(x,y)}{\partial y} &= -\frac{\partial B(x,y)}{\partial x},
\end{aligned}$$

which are often summarized as:

$$\frac{\partial f(z)}{\partial y} = i \frac{\partial f(z)}{\partial x} \cdot$$

Since the Jacobian determinant is $|f'(z)|^2$, the local inversion theorem allows us to deduce:

Corollary 2.4.3 *At all points of the disk of convergence such that $f('z) \neq 0$, the map f is locally invertible.*

Later, we shall prove that, if f is not constant, the zeroes of f are isolated. With the corollary above, this implies:

Corollary 2.4.4 *If f is not constant, it defines an open map on $\overset{\circ}{D}(0,r)$. (This means that it transforms open subsets of $\overset{\circ}{D}(0,r)$ into open sets.)*

Exercice 2.4.5 Let $k \geq 2$. Prove using the above corollaries that there exists an open disk U and a power series f defined on U such that $f^k = 1+z$. Deduce from that that, for $g \in \mathbf{C}\{z\}$ to be the k^{th} power of a power series, it is necessary and sufficient that $v_0(g)$ is a multiple of k.

2.5 Expansion of a power series at a point $\neq 0$

Let $f := \sum\limits_{n\geq 0} a_n z^n$ with r.o.c. $r > 0$ and let $z_0 \in \overset{\circ}{D}(0,r)$. We compute formally the expansion of f near z_0 as follows:

$$\begin{aligned}
f(z_0+z) &= \sum_{n\geq 0} a_n (z_0+z)^n \\
&= \sum_{n\geq 0} a_n \sum_{l+m=n} \frac{(l+m)!}{l!m!} z_0^l z^m \\
&= \sum_{m\geq 0} \left(\sum_{l\geq 0} \frac{(l+m)!}{l!m!} a_{l+m} z_0^l \right) z^m \\
&= \sum_{m\geq 0} \frac{f^{(m)}(z_0)}{m!} z^m,
\end{aligned}$$

since we already know that $f^{(m)}(z_0) = \sum\limits_{l\geq 0} \dfrac{(l+m)!}{l!} a_{l+m} z_0^l$. This calcultion can be rigorously justified, and one can prove (see for example the book of Cartan):

Theorem 2.5.1 *The Taylor series $\sum\limits_{m\geq 0} \dfrac{f^{(m)}(z_0)}{m!} z^m$ of the function f at z_0 is convergent. Its r.o.c. is at least equal to $r - |z_0|$. The function $g(z)$ it defines is equal to $f(z_0+z)$ on $\overset{\circ}{D}(0, r-|z_0|)$.*

□

Example 2.5.2 Let $f(z) = 1 + z + z^2 + \cdots = \dfrac{1}{1-z}$ (for $|z| < 1$) and let $|z_0| < 1$. Then:

$$f^{(n)}(z) = \sum_{k\geq 0} \frac{(n+k)!}{k!} z^k = \frac{n!}{(1-z)^{n+1}}.$$

Therefore:
$$\sum_{n\geq 0} \frac{f^{(n)}(z_0)}{n!} z^n = \sum_{n\geq 0} \frac{z^n}{(1-z_0)^{n+1}}.$$
This has r.o.c. $|1-z_0|$. For $|z| < |1-z_0|$, one has:
$$\sum_{n\geq 0} \frac{z^n}{(1-z_0)^{n+1}} = \frac{1}{1-z_0} \frac{1}{1-\frac{z}{1-z_0}} = \frac{1}{1-z_0-z} = f(z_0+z).$$
The very last equality makes sense because $|z| < |1-z_0| \Rightarrow |z_0+z| < 1$, so that z_0+z is indeed in the disk of convergence of f.

Exercice 2.5.3 In the example above, for what values of z_0 is the r.o.c. of the new power series bigger than $|1-z_0|$? Draw the corresponding disk to see how the domain of f has been extended.

Generally speaking, calling r' the new r.o.c., either $\overset{\circ}{\mathrm{D}}(z_0,r')$ goes beyond the boundary of $\overset{\circ}{\mathrm{D}}(0,r)$, or not. The points of the circle of indeterminacy which cannot be outcrossed in this way are "boundary points". It can be proved that there are always boundary points on the circle of indeterminacy. (This is a consequence of "Cauchy theory"). For some special power series, like $\sum\limits_{n\geq 0} z^{2^n}$, "Hadamard's theorem on lacunary series"implies that all the points of the circle of indeterminacy are boundary points.

2.6 Power series with values in a linear space

Let $V := \mathbf{C}^d$ and let $X(z) := \begin{pmatrix} f_1(z) \\ \vdots \\ f_d(z) \end{pmatrix}$, where the $f_i \in \mathbf{C}\{z\}$ have r.o.c. r_i. The vector-valued function X is defined and continuous on $\overset{\circ}{\mathrm{D}}(0,r)$, where $r := \min(r_1,\dots,r_d) > 0$.

Defining the $\mathbf{C}$-derivative of $X(z)$ as $X'(z) := \lim\limits_{h\to 0} \dfrac{X(z+h)-X(z)}{h}$ (also written $dX(z)/dz$), we see that it is indeed $\mathbf{C}$-derivable on $\overset{\circ}{\mathrm{D}}(0,r)$ and that $X'(z) = \begin{pmatrix} f_1'(z) \\ \vdots \\ f_d'(z) \end{pmatrix}$.

We can also group the power series expansions $f_i(z) = \sum a_{i,n} z^n$ in the form $X = \sum X_n z^n$, where $X_n := \begin{pmatrix} a_{1,n} \\ \vdots \\ a_{d,n} \end{pmatrix} \in V$. One can prove that, if for an arbitrary norm on V, one has $\lim\limits_{n\to+\infty} \sqrt[n]{||X_n||} = l$, then the r.o.c. of $X(z)$ is $1/l$.

The derivation of vector-valued functions is $\mathbf{C}$-linear: $(\lambda.X)' = \lambda.X'$ and $(X+Y)' = X'+Y'$. The Leibniz rule takes the form: $(f.X)' = f.X + f.X'$.

Exercice 2.6.1 Define matrix-valued functions $A(z) := (a_{i,j}(z))_{1\leq i,j\leq d}$ taking values in $\mathrm{Mat}_d(\mathbf{C})$, such that all $a_{i,j} \in \mathbf{C}\{z\}$. Write their $\mathbf{C}$-derivatives, the associated rules, the power series expansions. With A and X as described, what can be said of AX ?

Chapter 3

Analytic functions

3.1 Analytic and holomorphic functions

Definition 3.1.1 (i) Let f be a function or a germ. We say that f *admits a power series expansion at* $z_0 \in \mathbf{C}$ if there is a (convergent) power series $\sum\limits_{n \geq 0} a_n z^n \in \mathbf{C}\{z\}$ such that, for z in some neighborhood of 0, one has: $f(z_0 + z) = \sum\limits_{n \geq 0} a_n z^n$. We shall then rather write that, for z in some neighborhood of z_0, one has: $f(z) = \sum\limits_{n \geq 0} a_n (z - z_0)^n \in \mathbf{C}\{z - z_0\}$. For conciseness, we shall say that f is PSE ("power series expandable") at z_0.
(ii) Let f be a function on an open set $\Omega \subset \mathbf{C}$. We say that f *is analytic on* Ω if f admits a power series expansion at all points $z_0 \in \Omega$. An *analytic germ* is the germ of an analytic function.
(iii) A function analytic on the whole of $\mathbf{C}$ is said to be *entire*.

Examples 3.1.2 (i) The function $e^z = \sum\limits_{n \geq 0} \dfrac{e^{z_0}}{n!}(z - z_0)^n$ is PSE at any $z_0 \in \mathbf{C}$, so it is entire.
(ii) The function $\dfrac{1}{z} = \dfrac{1}{z_0(1 + \frac{z - z_0}{z_0})} = \sum\limits_{n \geq 0} \dfrac{(-1)^n}{z_0^{n+1}}(z - z_0)^n$ is PSE at any $z_0 \neq 0$, therefore it is analytic on $\mathbf{C}^*$. However, no power series describes it on the whole of $\mathbf{C}^*$.
(iii) If f is PSE at z_0 with a r.o.c. r, then it is analytic on $\overset{\circ}{\mathrm{D}}(z_0, r)$ (theorem 2.5.1).

Definition 3.1.3 (i) We say that the function or germ f is $\mathbf{C}$*-derivable at* z_0 if the limit $\lim\limits_{h \to 0} \dfrac{f(z_0 + h) - f(z_0)}{h}$ exists. This limit is called *the* $\mathbf{C}$*-derivative of* f *at* z_0 and written $f'(z_0)$ or $\dfrac{df}{dz}(z_0)$. From now on, we shall simply say "derivable, derivative" instead of "$\mathbf{C}$-derivable, $\mathbf{C}$-derivative".
(ii) A function f defined on an open set $\Omega \subset \mathbf{C}$ is said to be *holomorphic on* Ω if it is $\mathbf{C}$-derivable at every point of Ω. We then write f' or df/dz the function $z_0 \mapsto f'(z_0)$.

If we identify f with a function $F(x, y) = (A(x, y), B(x, y))$ (with real variables and with values in $\mathbf{R}^2$), then a necessary and sufficient condition for f to be holomorphic is that F be differentiable and that it satisfies the Cauchy-Riemann conditions:

$$\frac{\partial A(x,y)}{\partial x} = \frac{\partial B(x,y)}{\partial y} \text{ and } \frac{\partial A(x,y)}{\partial y} = -\frac{\partial B(x,y)}{\partial x} \text{ or, in a more compact form: } \frac{\partial f(z)}{\partial y} = \mathrm{i}\frac{\partial f(z)}{\partial x}\cdot$$

Then the Jacobian matrix is that of a direct similitude[1]:

$$JF(x,y) = \begin{pmatrix} \dfrac{\partial A(x,y)}{\partial x} & \dfrac{\partial A(x,y)}{\partial y} \\ \dfrac{\partial B(x,y)}{\partial x} & \dfrac{\partial B(x,y)}{\partial y} \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(f'(z)) & -\mathrm{Im}(f'(z)) \\ \mathrm{Im}(f'(z)) & \mathrm{Re}(f'(z)) \end{pmatrix}$$

Remark 3.1.4 The geometric consequence is that a non constant holomorphic function preserves the angles between tangent vectors of curves, and also the orientation: it is *conformal*.

Theorem 3.1.5 (FUNDAMENTAL !) *Analyticity and holomorphicity are equivalent properties.*

Proof. - The fact that an analytic function is holomorphic has been proved in the previous chapter. The converse implication will be admitted: see the books of Ahlfors, Cartan, Rudin. □

Some basic properties.

1. Analytic functions on an open set $\Omega \subset \mathbf{C}$ form a $\mathbf{C}$-algebra, which we write $O(\Omega)$.
2. If $f \in O(\Omega)$, then $1/f \in O(\Omega')$ where $\Omega' := \Omega \setminus f^{-1}(0)$. In particular, the elements of $O(\Omega)^*$ are the functions $f \in O(\Omega)$ which vanish nowhere.
3. If $f \in O(\Omega)$, $g \in O(\Omega')$ and $f(\Omega) \subset \Omega')$, then $g \circ f \in O(\Omega)$.
4. Let $z_0 \in \Omega$ and let δ denote the distance of z_0 to the exterior of Ω (or to its boundary, it is the same): $\delta := d(z_0, \mathbf{C} \setminus \Omega) = d(z_0, \partial\Omega) > 0$. Then f is indefinitely derivable at z_0, and equal to its Taylor series expansion $\sum\limits_{m \geq 0} \dfrac{f^{(m)}(z_0)}{m!}(z - z_0)^m$ on $\overset{\circ}{\mathrm{D}}(z_0, \delta)$ (which means implicitly that this series has r.o.c. $\geq \delta$).

The following theorem will play a central role in our course. We admit it (see the books by Ahlfors, Cartan, Rudin).

Theorem 3.1.6 (Principle of analytic continuation) *Suppose that Ω is a domain (a connected open set). If $f \in O(\Omega)$ vanishes on a non empty open set, then $f = 0$ (the zero function on Ω).*

□

As a consequence, if f is not the zero function on Ω, at every $z_0 \in \Omega$ it has a non trivial power series expansion: $f(z) = \sum\limits_{n \geq k} a_n (z - z_0)^n$, with $a_k \neq 0$. Then $f = (z - z_0)^k g$, where g is PSE at z_0 and $g(z_0) \neq 0$. We shall then write $v_{z_0}(f) = k$.

This implies in particular that, in some neighborhood of z_0, g does not vanish.

Corollary 3.1.7 *The zeroes of a non trivial analytic function on a domain are isolated.*

[1] Remember that a direct similitude in the real plane $\mathbf{R}^2$ has a matrix of the form $\begin{pmatrix} u & -v \\ v & u \end{pmatrix}$. It corresponds in the complex plane $\mathbf{C}$ to the map $z \mapsto wz$, where $w := u + \mathrm{i}v$. These are the only linear maps that preseerve angles and orientation.

Remember from the previous chapter that, if f is analytic on a domain Ω , then, at all points $z_0 \in \Omega$ such that $f'(z_0) \neq 0$, the map f is locally invertible. With the corollary above, this implies:

Corollary 3.1.8 *If f is not constant, it defines an open map on $\overset{\circ}{D}(0,r)$. (This means that it transforms open subsets of $\overset{\circ}{D}(0,r)$ into open subsets of $\mathbf{C}$.)*

Corollary 3.1.9 *Let f be a non trivial analytic function on a domain. Then, on every compact set, f has finitely many zeroes. Altogether, the set $f^{-1}(0)$ of its zeroes is at most denumerable.*

Exercice 3.1.10 Find the zeroes of $\sin(\pi/z)$. Do they accumulate ? Does this contradict the above results ?

3.2 Singularities

Theorem 3.2.1 (Riemann's theorem of inexistent singularities) *Let $\Omega \subset \mathbf{C}$ be an open set and let $z_0 \in \Omega$. Assume that $f \in \mathcal{O}(\Omega \setminus \{0\})$ is bounded on some punctured neighborhood of z_0, that is on some $U \setminus \{0\}$, where U is a neighborhood of z_0. Then f admits a continuation at z_0 which makes it an analytic function on the whole of Ω.*

Proof. - See the books of Ahlfors, Cartan, Rudin. □

Obviously, the said continuation is unique and we shall identify it with f (and write it f).

Example 3.2.2 The function $f(z) := \dfrac{z}{e^z - 1}$ is analytic on $\Omega := \mathbf{C} \setminus 2i\pi\mathbf{Z}$. But since $\lim\limits_{z \to 0} f(z) = \dfrac{1}{\exp'(0)} = 1$, the function f is bounded near 0 and can be continuated there by putting $f(0) := 1$.

Exercice 3.2.3 Show that the power series expansion of f at 0 has r.o.c. 2π. Give a way to compute recursively its coefficients and find them up to degree 6.

Corollary 3.2.4 *Let $\Omega \subset \mathbf{C}$ be an open set and let $z_0 \in \Omega$ and $f \in \mathcal{O}(\Omega \setminus \{0\})$. Three cases are possible:*

1. *If f is bounded on some punctured neighborhood of z_0, we consider it as analytic on Ω.*

2. *Else, if there exists $N \geq 1$ such that $f(z) = O\left(|z - z_0|^N\right)$, then there exists a unique $k \geq 1$ such that $g := (z - z_0)^k f$ is analytic and $g(z_0) \neq 0$. In this case, f is said to have* a pole of order k at z_0. *We put $v_{z_0}(f) := -k$.*

3. *Else, we say that f* has an essential singularity at z_0.

Example 3.2.5 The function $\dfrac{e^{1/z}}{e^z - 1}$ has simple poles (*i.e.* of order 1) at all points of $2i\pi\mathbf{Z}$ except 0 and an essential singularity at 0.

If f has a pole of order k at z_0, it admits a (convergent) Laurent series expansion $f(z) = \sum_{n\geq -k} a_n(z-z_0)^n$, with $a_{-k} \neq 0$. We write $\mathbf{C}(\{z-z_0\})$ the $\mathbf{C}$-algebra of such series. It is the field of fractions of $\mathbf{C}\{z-z_0\}$ and it is actually equal to $\mathbf{C}\{z-z_0\}[1/(z-z_0)]$. Clearly, poles are isolated.

On the other hand, in the case of an essential singularity, f has a "generalized Laurent series expansion", with infinitely many negative powers; for instance, $e^{1/z} = \sum_{n\geq 0} z^{-n}/n!$. Essential singularities need not be isolated, as shows the example of $1/\sin(1/z)$ at 0.

Definition 3.2.6 The function f is said to be meromorphic on the open set Ω if there is a discrete subset $X \subset \Omega$ such that f is analytic on $\Omega \setminus X$ and has poles on X.

The following is easy to prove:

Theorem 3.2.7 *Meromorphic functions on a domain Ω form a field $\mathcal{M}(\Omega)$. In particular, if $f,g \in \mathcal{O}(\Omega)$ and $g \neq 0$, then $f/g \in \mathcal{M}(\Omega)$.*

Much more difficult is the theorem (due to Hadamard) that all meromorphic functions are quotients of holomorphic functions.

Examples 3.2.8 (i) Rational functions $f := P/Q \in \mathbf{C}(z)$ are meromorphic on $\mathbf{C}$. If $P,Q \in \mathbf{C}[z]$ are coprime polynomials and if z_0 is a root of order k of P, then $v_{z_0}(f) = k$. If z_0 is a root of order k of Q, then it is a pole of order k of f and $v_{z_0}(f) = -k$. If z_0 is a root of neither P nor Q, then $v_{z_0}(f) = 0$.
(ii) If Ω is a domain and $f \in \mathcal{M}(\Omega)$, $f \neq 0$, then $f'/f \in \mathcal{M}(\Omega)$. The poles of f'/f are the zeroes and the poles of f. They are all simple. All this comes from the fact that if $f := (z-z_0)^k g$ with g analytic at z_0 and $g(z_0) \neq 0$, then $f'/f = k/(z-z_0) + g'/g$, and g'/g is analytic at z_0.

3.3 Cauchy theory

Let Ω be a domain, $f \in \mathcal{O}(\Omega)$ and $\gamma : [a,b] \to \Omega$ be a continuous path ($a,b \in \mathbf{R}$ and $a < b$). We shall define:

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

For this definition to make sense, we shall require the path γ to be of class $\mathcal{C}^1$, that is, continuously differentiable. Note however that the weaker assumption: piecewise continuously differentiable (and continuous) would be sufficient. One can check easily that, in the above formula, reparameterizing the path (that is, using $\gamma(\phi(s)$ where $\phi : [a',b'] \to [a,b]$ is a change of parameters) does not change the integral.

Note that, if f has a primitive F in Ω (that is $F \in \mathcal{O}(\Omega)$ and $F' = f$), then $\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$. In particular, if γ is a loop, then $\int_\gamma f(z)\,dz = 0$.

Examples 3.3.1 (i) Let $k \in \mathbf{Z}$ and $\gamma_1 : [0,2\pi] \to \mathbf{C}^*$, $t \mapsto e^{it}$ and $\gamma_2 : [0,1] \to \mathbf{C}^*$, $t \mapsto e^{2i\pi t}$. Let $f(z) := z^n$, $n \in \mathbf{Z}$. Then:

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz = \begin{cases} 2i\pi k \text{ if } n = -1, \\ 0 \text{ other wise.} \end{cases}$$

Indeed:
$$\int_{\gamma_1} z^n\,dz = \int_0^{2\pi} \mathrm{i}ke^{\mathrm{i}k(n+1)t}\,dt \text{ and } \int_{\gamma_2} z^n\,dz = \int_0^1 2\mathrm{i}\pi ke^{2\mathrm{i}\pi k(n+1)t}\,dt.$$
(ii) From this, by elementary computation, one finds that if f has a generalized Laurent series expansion (be it holomorphic, meromorphic or essentially singular) $\sum\limits_{n\in\mathbf{Z}} a_n z^n$ at 0, and if its domain of existence contains the unit circle, then:
$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz = 2\mathrm{i}\pi k a_{-1}.$$

The following important theorems are proved in the books by Ahlfors, Cartan and Rudin.

Theorem 3.3.2 (Cauchy) *If $f \in O(\Omega)$ and $\gamma_1, \gamma_2 : [a,b] \to \Omega$ are two homotopic pathes of class C^1 (that is, they can be continuously deformed into each other within Ω), then:*
$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

□

Using the calculations in the examples, one deduces:

Corollary 3.3.3 *If $f(z) = \sum\limits_{n\in\mathbf{Z}} a_n(z-z_0)^n$ and if the loop γ has its image in the domain of existence of f, then:*
$$\int_{\gamma} f(z)\,dz = 2i\pi I(z_0,\gamma)a_{-1}.$$

(Remember that the index $I(z_0,\gamma)$ is the number of times that the loop γ turns around z_0 in the positive sense.)

Definition 3.3.4 If $f(z) = \sum\limits_{n\in\mathbf{Z}} a_n(z-z_0)^n$, the complex number a_{-1} is called the *residue of f at z_0* and written $\mathrm{Res}_{z_0}(f)$.

Theorem 3.3.5 (Cauchy residue formula) *Let f have a finite number of singularities in Ω and let γ be a loop in Ω avoiding all these singularities. Then:*
$$\int_{\gamma} f(z)\,dz = 2i\pi \sum_{z_0} I(z_0,\gamma) Res_{z_0}(f),$$

the sum being taken for all singularities z_0.

□

Exercice 3.3.6 Let $\gamma(t) := Re^{\mathrm{i}t}$ on $[0,2\pi]$, where R is "big" and $R \notin 2\mathrm{i}\pi\mathbf{N}$. For $k \in \mathbf{N}$, compute $\int_{\gamma} z^{-k} \dfrac{z}{e^z - 1}\,dz$.

Corollary 3.3.7 *Suppose $f \in \mathcal{M}(\Omega)$ (a domain), $f \neq 0$, and γ is a loop in Ω avoiding all zeroes and poles of f. Then:*

$$\int_\gamma (f'/f)(z)\,dz = 2i\pi \sum_{z_0} I(z_0,\gamma) v_{z_0}(f),$$

the sum being taken for all zeroes and poles.

Corollary 3.3.8 (Cauchy formula) *Suppose $f \in \mathcal{O}(\Omega)$ and γ is a loop in Ω avoiding $z_0 \in \Omega$. Then:*

$$\int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\,dz = 2i\pi I(z_0,\gamma)\frac{f^{(k)}(z_0)}{k!}.$$

Primitives Curvilinear integrals (*i.e.*integrals on pathes) can serve to compute primitives (or prove they do not exist: see remark below and next chapter for this). Let Ω be a simply connected domain, that is, all loops can be continuously shrinked to a point. Then by Cauchy theorem on homotopy invariance, if $f \in \mathcal{O}(\Omega)$ and $\gamma_1, \gamma_2 : [a,b] \to \Omega$ are any two pathes of class $\mathcal{C}^1$, we have:

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

Now fix $a = z_0$ and consider $b = z$ as a variable. The the above integrals define a same function $F(z)$ on Ω.

Theorem 3.3.9 *This function is the unique primitive of f (that is, $F' = f$) such that $F(z_0) = 0$.*

$\square$

Remark 3.3.10 If Ω is not simply connected, some functions may have no primitives on Ω. For instance, if $a \notin \Omega$ and there is a loop γ in Ω such that $I(a,\gamma) \neq 0$, then the function $f(z) := \dfrac{1}{z-a}$ is holomorphic on Ω but has no primitive; indeed, $\int_\gamma f(z)\,dz = I(a,\gamma)$.

Chapter 4

The complex logarithm

4.1 Can one invert the complex exponential function ?

We know that the real exponential function $x \mapsto e^x$ from $\mathbf{R}$ to $\mathbf{R}_+^*$ can be inverted by $\ln : \mathbf{R}_+^* \to \mathbf{R}$ in the sense that $\forall x \in \mathbf{R}$, $\ln(e^x) = x$ and $\forall y \in \mathbf{R}_+^*$, $e^{\ln(y)} = y$. Moreover, ln is a rather "good" function: it is continuous, derivable, etc. We are going to *try* to extend this process to $\mathbf{C}$, that is, to invert the complex exponential function $\exp : \mathbf{C} \to \mathbf{C}^*$. (However, we shall keep the notation ln for the real logarithm mentioned above.) It is impossible to have a function $L : \mathbf{C} \to \mathbf{C}^*$ such that $\forall z \in \mathbf{C}$, $L(e^z) = z$. Indeed, since $e^{z+2\mathrm{i}\pi} = e^z$, this would imply $z + 2\mathrm{i}\pi = z$. Clearly, the impossibility stems from the fact that exp is not injective. However, we know that exp is surjective, so that for each $z \in \mathbf{C}^*$ there exists a (non unique) complex number, say $L(z)$, such that $e^{L(z)} = z$. In this way, we can build a function $L : \mathbf{C}^* \to \mathbf{C}$ such that $\forall z \in \mathbf{C}^*$, $e^{L(z)} = z$. Now, the values $L(z)$ having been choxen at random (each time among infinitely many choices), it is not clear that one can get in this way a "good" function. Indeed one cannot:

Lemma 4.1.1 *There is no continuous function* $L : \mathbf{C}^* \to \mathbf{C}$ *such that* $\forall z \in \mathbf{C}^*$, $e^{L(z)} = z$.

Proof. - Actually, it is not even possible such a continuous function on $\mathbf{U}$. Assume indeed there was one such function $L : \mathbf{U} \to \mathbf{C}$ and put, for all $t \in \mathbf{R}$, $f(t) := L(e^{\mathrm{i}t}) - \mathrm{i}t$. Then f is a continuous function from $\mathbf{R}$ to $\mathbf{C}$. Since $e^{L(e^{\mathrm{i}t})} = e^{\mathrm{i}t}$, we see that $e^{f(t)} = 1$ for all t. Therefore the continuous function f sends the connected set $\mathbf{R}$ to the discrete set $2\mathrm{i}\pi\mathbf{Z}$; this is only possible if it is constant. Thus, there exists a fixed $k \in \mathbf{Z}$ such that: $\forall t \in \mathbf{R}$, $L(e^{\mathrm{i}t}) = \mathrm{i}t + 2\mathrm{i}\pi k$. Now, writing this for t and $t + 2\pi$ yields the desired contradiction. □

Therefore, we are going to look for *local determinations of the logarithm*: this means a continuous function $L : \Omega \to \mathbf{C}$, where Ω is some open subset of $\mathbf{C}^*$, such that $\forall z \in \Omega$, $e^{L(z)} = z$. This will not be possible for arbitrary Ω.

Lemma 4.1.2 *Let* $\Omega \subset \mathbf{C}^*$ *a domain (a connected open set). Then any two determinations of the logarithm on* Ω *differ by a constant. (Of course, there may exist no such determination at all !)*

Proof. - If L_1 and $L2$ are two determinations of the logarithm on Ω, then $\forall z \in \Omega$, $e^{L_2(z) - L_1(z)} = 1$, so that the continuous function $L_2 - L_1$ sends the connected set Ω to the discrete set $2\mathrm{i}\pi\mathbf{Z}$, so it is constant. □

Therefore, if there is at least one determination of the logarithm on Ω, there is a denumerable family of them differing by constants $2i\pi k$, $k \in \mathbf{Z}$. If one wants to specify one of them, one uses an *initial condition*: for some $z_0 \in \Omega$, one chooses a particular $w_0 \in \mathbf{C}$ such that $e^{w_0} = z_0$, and one knows that there is a unique determination such that $L(z_0) = w_0$.

4.2 The complex logarithm via trigonometry

We fix $\theta_0 \in \mathbf{R}$ arbitrary, indicating a direction in the place, that is a half-line $\mathbf{R}_+ e^{i\theta_0}$. We define the "cut plane" $\Omega := \mathbf{C} \setminus \mathbf{R}_- e^{i\theta_0}$ (that is, we think that we have "cut" the "prohibited half-line" $\mathbf{R}_- e^{i\theta_0}$); it is an open subset of $\mathbf{C}^*$. Then, for all $z \in \Omega$, there is a unique pair $(r, \theta) \in \mathbf{R}_+^* \times]\theta_0 - \pi, \theta_0 + \pi[$ such that $z = re^{i\theta}$. Moreover, r and θ are continuous functions of z. Therefore, putting $L_{\theta_0}(z) := \ln(r) + i\theta$, we get a continuous function $L_{\theta_0} : \Omega \to \mathbf{C}$. This is clearly a determination of the logarithm on Ω, characterized by the initial condition $L_{\theta_0}(e^{i\theta_0}) = i\theta_0$.

If we take $\theta_0 := 0$, we get the *principal determination of the logarithm*, which we write log. It is defined on the cut plane $\mathbf{C} \setminus \mathbf{R}_- \subset \mathbf{C}^*$ and characterized by the initial condition $\log(1) = 0$. Its restriction to $\mathbf{R}_+^*$ is ln.

Remark 4.2.1 This nice function cannot be continuously extended to $\mathbf{R}_-$. Indeed, if for instance $z \in \mathbf{C} \setminus \mathbf{R}_-$ tends to -1, then it can be written $z = re^{i\theta}$ where $r > 0$ and $-\pi < \theta < \pi$. One has $r \to 1$ and $\theta \to \pm\pi$: if z approaches -1 by above, then $\theta \to +\pi$; if z approaches -1 by below, then $\theta \to -\pi$. In the first case, $\log(z) \to i\pi$; in the second case, $\log(z) \to -i\pi$. In full generality, z could alternate above and below and then $\log(z)$ would tend to nothing.

Remark 4.2.2 If we change the argument θ_0 by $\theta_0 + 2\pi$, the open set Ω does not change, but the determination of the logarithm does. We know that L_{θ_0} and $L_{\theta_0+2\pi}$ differ by a constant, so we just have to test them on the initial conditions. Since $L_{\theta_0}(e^{i\theta_0}) = i\theta_0$ and $L_{\theta_0+2\pi}(e^{i(\theta_0+2\pi)}) = i(\theta_0 + 2\pi)$, and since $e^{i(\theta_0+2\pi)} = e^{i\theta_0}$, we conclude that $L_{\theta_0+2\pi} = L_{\theta_0} 2i\pi$.

The following series of exercices is long, because it is important to get familiar with the strange behaviour of the complex logarithm.

Exercice 4.2.3 Under what condition on θ_0 does the open set Ω contain the positive real half-line $\mathbf{R}_+^*$? Assuming this, under which supplementary condition does one have $L_{\theta_0}(1) = 0$? Assuming again this, show that the restriction of L_{θ_0} to $\mathbf{R}_+^*$ is ln.

Exercice 4.2.4 (i) When does one have $\log(e^z) = z$?
(ii) Compare $\log(ab)$ with $\log(a) + \log(b)$.

Exercice 4.2.5 (i) Suppose $\theta \in]\theta_0 - \pi, \theta_0 + \pi[$ and also $\theta \in]\theta_1 - \pi, \theta_1 + \pi[$. Then compare $L_{\theta_0}(e^{i\theta})$ with $L_{\theta_1}(e^{i\theta})$.
(ii) Let $\Omega_0 := \mathbf{C} \setminus \mathbf{R}_- e^{i\theta_0}$ and $\Omega_1 := \mathbf{C} \setminus \mathbf{R}_- e^{i\theta_1}$. Describe the intersection $\Omega_0 \cap \Omega_1$ and compare L_{θ_0} with L_{θ_1} on this set.

4.3 The complex logarithm as an analytic function

Let $L(z) := \sum\limits_{n\geq 1} \dfrac{(-1)^{n-1}}{n}(z-1)^n$. Then L is analytic on $\overset{\circ}{\mathrm{D}}(1,1)$ and satisfies the initial condition $L(1)=0$. Moreover, $L'(z) = \sum\limits_{n\geq 0} (-1)^n (z-1)^n = \dfrac{1}{z}$ on $\overset{\circ}{\mathrm{D}}(1,1)$, from which one draws $(e^L)' = e^L/z$, then $(e^L/z)' = 0$ and one concludes that $e^{L(z)} = z$. Therefore, L is a determination of the logarithm on $\overset{\circ}{\mathrm{D}}(1,1)$. Since $\overset{\circ}{\mathrm{D}}(1,1) \subset \mathbf{C} \setminus \mathbf{R}_-$, we deduce from the initial condition at 1 that L is the restriction of log (the principal determination) to $\overset{\circ}{\mathrm{D}}(1,1)$:

$$\forall z \in \overset{\circ}{\mathrm{D}}(1,1)\ ,\ \log(z) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n}(z-1)^n.$$

Proposition 4.3.1 *Let $z_0 := re^{i\theta_0}$ and $w_0 := \ln(r) + i\theta_0$. Then:*

$$\forall z \in \overset{\circ}{D}(z_0,|z_0|)\ ,\ L_{\theta_0}(z) = w_0 + \sum_{n\geq 1} \frac{(-1)^{n-1}}{nz_0^n}(z-z_0)^n.$$

Proof. - The function $M(z) := w_0 + L(z/z_0)$ is analytic on $\overset{\circ}{\mathrm{D}}(z_0,|z_0|)$ and satisfies the relations: $M(z_0) = w_0$ and, since w_0 is a logarithm[1] of z_0, $e^{M(z)} = z$. Therefore it is the (unique) determination of the logarithm on $\overset{\circ}{\mathrm{D}}(z_0,|z_0|)$ satisfying the same initial condition as L_{θ_0}. □

Corollary 4.3.2 *All determinations of the logarithm are analytic functions.*

Proof. - Indeed, at any point, there is one determination on some disk which is an analytic function (the one in the proposition); and they all differ by constants. □

Primitives. We have already used the following argument: $L' = 1/z \Longrightarrow (e^L)' = e^L/z \Longrightarrow (e^L/z)' = 0$. Therefore, if $L' = 1/z$ on a domain $\Omega \subset \mathbf{C}^*$ and, for some $z_0 \in \Omega$, $L(z_0)$ is a logarithm of z_0, then L is a determination of the logarithm on Ω. Therefore, we can define L in the following way.

Proposition 4.3.3 *Let Ω be a domain in $\mathbf{C}^*$ such that for any loop γ in Ω one has $I(0,\gamma) = 0$. (This is true for instance if Ω is simply connected.) Fix z_0 in Ω and fix w_0 a logarithm of z_0. Then the function:*

$$L(z) := w_0 + \int_\gamma \frac{dz}{z},$$

where γ is any path from z_0 to z, is well defined and it is the unique determination of the logarithm on Ω satisfying the initial condition $L(z_0) = w_0$.

□

Exercice 4.3.4 Prove that conversely, if there is a determination of the logarithm on Ω, then for any loop γ in Ω one has $I(0,\gamma) = 0$.

[1] We say *a* logarithm of $z \in \mathbf{C}^*$ for a complex number w such that $e^w = z$; there are infinitely many logarithms of z. We shall try not to cause confusion with the "determinations of *the* logarithm", which are functions.

4.4 The logarithm of an invertible matrix

We know that, if $A \in \mathrm{Mat}_n(\mathbf{C})$, then $\exp(A) \in \mathrm{GL}_n(\mathbf{C})$. We shall now prove that the exponential map $\exp : \mathrm{Mat}_n(\mathbf{C}) \to \mathrm{GL}_n(\mathbf{C})$ is surjective. This fact will be needed in the sequel, but not its proof: the reader may skip it if he wants to. But the proof is not easily found in books, so I still prefer to give it.

Semi-simple matrices. On $\mathbf{C}$, these are the same thing that diagonalizable matrices; but the notion is more general, and this terminology is more similar to that of algebraic groups, that we shall use later. So we consider a matrix $S := P\mathrm{Diag}(\mu_1,\dots,\mu_n)P^{-1} \in \mathrm{GL}_n(\mathbf{C})$. Since S is invertible, its eigenvalues are non zero: $\mu_i \in \mathbf{C}^*$, and we can choose logarithms $\lambda_i \in \mathbf{C}$ such that $e^{\lambda_i} = \mu_i$ for $i = 1,\dots,n$. Then, $A := P\mathrm{Diag}(\lambda_1,\dots,\lambda_n)P^{-1}$ satisfies $\exp(A) = S$.

However, for technical reasons that will appear in the proof of the theorem below, we want more. So we make a refined choice of the logarithms λ_i. Precisely, we choose them so that, whenever $\mu_i = \mu_j$ then $\lambda_i = \lambda_j$. It is then not hard, using poynomial interpolation (for instance, Lagrange interpolation polynomials) to see that there exists a polynomial $F \in \mathbf{C}[z]$ such that $F(\mu_i) = \lambda_i$ for $i = 1,\dots,n$. Then one draws that $A = F(S)$.

Unipotent matrices. Let $U \in \mathrm{GL}_n(\mathbf{C})$ be such that $U - I_n$ is nilpotent; then we know that $(U - I_n)^n = 0_n$. Define $N := \sum\limits_{1 \leq k < n} \dfrac{(-1)^{k-1}}{k}(U - I_n)^k$. Then $\exp(N) = U$. Indeed, this follows from the composition of the formal series for exp and log, except that here we truncated the latter, taking off the vanishing terms. Note that again the "logarithm" N of U is obtained in the form $G(U)$ for some polynomial $G \in \mathbf{C}[z]$.

Jordan decomposition. It is a classical fact in linear algebra that every matrix $M \in \mathrm{Mat}_n(\mathbf{C})$ admits a unique decomposition in the form $M = M_s + M_n$, where M_s is semi-simple, M_n is nilpotent and they commute: $M_sM_n = M_nM_s$. This is sometimes called the Dunford decomposition. Moreover, M_s has the same spectrum as M. Therefore, we take $M \in \mathrm{GL}_n(\mathbf{C})$, then $M_s \in \mathrm{GL}_n(\mathbf{C})$ and we can write $M = M_sM_u = M_uM_s$, where $M_u := I_n + M_s^{-1}M_n = I_n + M_nM_s^{-1}$ is unipotent. This is the *Jordan decomposition of M*; it is of course also unique. (In France this is rather called "multiplicative Dunford decomposition").

Theorem 4.4.1 *Let $B \in GL_n(\mathbf{C})$. Then there exists $A \in Mat_n(\mathbf{C})$ such that $\exp(A) = B$.*

Proof. - Write $B = B_sB_u$ the Jordan decomposition. Find a semi-simple matrix $A_s = F(B_s)$ such that $\exp(A_s) = B_s$ and a nilpotent matrix $A_n = G(B_u)$ such that $\exp(A_n) = B_n$, where $F, G \in \mathbf{C}[z]$ are polynomials. Last, define $A := A_s + A_n$. Then $B_sB_u = B_uB_s \Rightarrow F(B_s)G(B_u) = G(B_u)F(B_s)$, *i.e.* $A_sA_n = A_nA_s$, so that $\exp(A) = \exp(A_s + A_n) = \exp(A_s)\exp(A_n) = B_sB_u = B$. □

Note that $A := A_s + A_n$ is the Dunford decomposition of A.

Exercice 4.4.2 Prove that $\exp(A) = I_n$ is equivalent to: A is semi-simple and all its eigenvalues are in $2i\pi\mathbf{Z}$. Can such an A be upper triangular ?

Chapter 5

From the local to the global

5.1 Analytic continuation

We saw that there are various incarnations of the logarithm in various regions of the plane. This is a very general (and fundamental) phenomenon regarding analytic functions. We shall formalize it as a *process of analytic continuation along a path.*

The data.

1. Let $a \in \mathbf{C}$ and let f be analytic in some neighborhood of a. The neighborhood does not matter, so we consider f as a *germ* at a and write $f \in O_a = \mathbf{C}\{z-a\}$.
2. Let $b \in \mathbf{C}$ and let $\gamma : [0,1] \to \mathbf{C}$ be a path from $\gamma(0) = a$ to $\gamma(1) = b$. We require that γ be continuous, nothing more. Of course, we could take another interval as a source for γ.
3. Let $0 = t_0 < t_1 < \cdots < t_n = 1$ a *subdivision* of $[0,1]$.
4. We cover the image curve $\gamma([0,1]) \subset \mathbf{C}$ by open disks $D_i := \overset{\circ}{\mathrm{D}}(z_i, r_i)$, where, for $i = 0, \ldots, n$, we have $z_i = \gamma(t_i)$ (thus a point on the curve) and $r_i > 0$. Note that the first and last disk are respectively centered at $z_0 = \gamma(0) = a$ and at $z_n = \gamma(1) = b$. We assume that, for $i = 1, \ldots, n$ the disks D_i and D_{i-1} have a non empty intersection: $D_i \cap D_{i-1} \neq \emptyset$.

Definition 5.1.1 Suppose that there are functions $f_i \in O(D_i)$ for $i = 0, \ldots, n$ such that f is the germ of f_0 and that, for $i = 1, \ldots, n$, the functions f_i and f_{i-1} have the same restriction on $D_i \cap D_{i-1}$. Call $g \in O_b$ the germ of f_n. Then g is called *the result of the analytic continuation of f along γ.*

Note that, the data above being fixed, this result is necessarily unique. Indeed, from the principle of analytic continuation (theorem 3.1.6), and since the D_i are domains, f_0 is uniquely determined by its gem f; and, for $i = 1, \ldots, n$, each f_i is uniquely determined by its restriction to $D_i \cap D_{i-1}$, thus by f_{i-1}. Moreover, with some combinatorial and geometrical reasoning, one can see that the choice of the t_i, the z_i and the r_i does not change the result: that is why it is sound, in the definition, to speak of γ alone and not of the other data. Actually, the process is even much more invariant as shows the following essential result (the proof of which can be found in the book of Ahlfors).

Theorem 5.1.2 (Principle of monodromy) *Let Ω be a domain, $a, b \in \Omega$ and $f \in \mathcal{O}_a$ an analytic germ at a. Assume that, for all pathes from a to b in Ω, analytic continuation of f along γ is possible. Then, if γ_1 and γ_2 are two pathes from a to b which are homotopic in Ω (that is, they can be deformed into each other within Ω), then the result of the analytic continuation of f along γ_1 or γ_2 is the same.*

$\square$

Remark 5.1.3 Analytic continuation is not always possible: for instance, the germ at 0 defined by the lacunary series $\sum z^{2^n}$ admits no analytic continuation out of $\overset{\circ}{\mathrm{D}}(0,1)$.

Example 5.1.4 Let $f_0(z) := \sum\limits_{n \geq 0} \binom{1/2}{n}(z-1)^n = (1+(z-1))^{1/2}$, which, for obvious reasons, we write $\sqrt{z}$: it is an analytic function on $\overset{\circ}{\mathrm{D}}(1,1)$. It can be defined trigonometrically by the formula $f_0(re^{\mathrm{i}\theta}) = \sqrt{r}e^{\mathrm{i}\theta/2}$ for $-\pi < \theta < \pi$. (Argument: both functions are continuous on a domain, with same square and same initial value at 1.) Put $\gamma(t) = e^{\mathrm{i}t}$ on the segment $[0, 2\pi]$, thus a loop: the most interesting case ! Take $n = 4$ and the subdivision of the $t_k = k/4$ for $k = 0, \ldots, 4$, so that $z_k = \mathrm{i}^k$. Take all radii $r_k := 1$. We have $z_0 = z_4 = 1$, the base point of the loop; and the circle is covered by four disks, because the first and last disks are equal: $D_0 = D_4$.
Now we define functions similar to f_0 in the following way; the function f_k will be defined on D_k: $f_k(re^{\mathrm{i}\theta}) = \sqrt{r}e^{\mathrm{i}\theta/2}$ for $\theta \in]k\pi/4 - \pi, k\pi/4 + \pi[$. Thus, each f_k is a continuous determination of the square root on D_k. Thus, on each $D_k \cap D_{k-1}$ (which are non empty domains) the functions f_k and f_{k-1} are equal or opposite (because the quotient of the two functions is continuous with values in $\{+1, -1\}$). To check that they are equal, one initial condition is enough. It can be found each time by using the point $\mathrm{i}^{k-1}e^{\mathrm{i}\pi/4} \in D_k \cap D_{k-1}$.
The conclusion is that the germ g of f_4 at 1 is the analytic continuation of the germ f of f_0 at 1 along γ. But $f_4 = -f_0$: the square root $\sqrt{z}$ has been transformed into its opposite.

Exercice 5.1.5 If a domain $\Omega \subset \mathbf{C}^*$ contains the image of the above loop, show that there is no continuous function f on Ω such that $f(z)^2 = z$. (Consider the function $f(e^{\mathrm{i}t})e^{-\mathrm{i}t/2}$).

Example 5.1.6 We use the same loop, subdivision and disks than in the previous example and look for the analytic continuation of the germ at 1 of the function $f_0(z) := \sum\limits_{n \geq 1} \dfrac{(-1)^{n-1}}{n}(z-1)^n$, the principal determination of the logarithm. We know that $f_0(re^{\mathrm{i}\theta}) = \ln(r) + \mathrm{i}\theta$ for $-\pi < \theta < \pi$. We define f_k on D_k by $f_k(re^{\mathrm{i}\theta}) = \ln(r) + \mathrm{i}\theta$ for $\theta \in]k\pi/4 - \pi, k\pi/4 + \pi[$. These are also determinations of the logarithms, so in their common domains they differ by constants in $2\mathrm{i}\pi\mathbf{Z}$. Using the same points as before as initial conditions, we find that f_k and f_{k-1} are equal on $D_k \cap D_{k-1}$ and therefore that the germ g of f_4 at 1 is the analytic continuation of the germ f of f_0 at 1 along γ. But $f_4 = f_0 + 2\mathrm{i}\pi$: the logarithm $\log(z)$ has been transformed into $\log(z) + 2\mathrm{i}\pi$.

Exercice 5.1.7 Deduce from this a new proof that there is no determination of the logarithm on the whole of $\mathbf{C}^*$. (If there was, it would be equal to f_k on D_k.)

5.2 Monodromy

The principle of "monodromy", after the greek "mono" for unique and "dromos" for path, means that the result of analytic continuation does not depend on the path - except if something prevents deformation. We shall see that, for differential equations, what prevents deformations is usually the presence of singularities, and we shall consider monodromy as the effect of this singularities on the changes of values by analytic continuations. In some sense, we shall rather try to understand the multiplicity than the unicity !

Now, with some algebraic formalism, we shall give power to the principle of monodromy. It will be useful to have a notation[1] for the result of analytic continuation along a path. So if $f \in O_a$ and if the path γ goes from a to b, then we write f^γ the result of the analytic continuation of f along γ if it exists: thus the notation may represent nothing in some cases. Here are the successive steps of the algebraic formalisation. We suppose that a domain Ω has been fixed and everything (points, pathes, homotopies, arguments of functions ...) lives there. For $a \in \Omega$, we shall write $\tilde{O}_a$ the subset of O_a made of germs which admit analytic continuation along any path in Ω starting from a.

1. Suppose that $f, g \in O_a$ admit an analytical continuation along the path γ from a to b. Then adding them, multiplying them, derivating them yields the same relations and operations between intermediate functions, and therefore between the results:

$$
\begin{aligned}
(\lambda f + \mu g)^\gamma &= \lambda f^\gamma + \mu g^\gamma, \\
(fg)^\gamma &= f^\gamma g^\gamma, \\
(f')^\gamma &= (f^\gamma)'.
\end{aligned}
$$

   Therefore, the subset of O_a formed by germs which admit an analytic continuation along γ is a sub-$\mathbf{C}$-algebra of O_a and it is moreover stable under derivation: we say that it is a sub-*differential algebra* of O_a. Of course, $\tilde{O}_a$ is itself a sub-differential algebra of this differential algebra. Moreover, $f \mapsto f^\gamma$ is a morphism of differential algebras (it is linear, a morphism of rings, and it commutes with derivation). Altogether, these facts are called "principle of conservation of algebraic and differential relations". They are a natural property of monodromy, true for transcendental reasons (based on analysis) but their algebraization is the basis of Differential Galois Theory.

2. If γ_1 goes from a to b and γ_2 goes from b to c, then we write $\gamma_1.\gamma_2$ the composite path from a to c. Then, if $f \in O_a$ is continuated to $g \in O_b$ along γ_1 and g is continuated to $h \in O_c$ along γ_2, it is clear that f is continuated to h along $\gamma_1.\gamma_2$:

$$
f^{\gamma_1.\gamma_2} = (f^{\gamma_1})^{\gamma_2},
$$

   meaning that, if one side of the equality is meaningful, so is the other and then they are equal.

3. Let γ_1 and γ_2 be two pathes from a to b and suppose that they are homotopic (in Ω by convention). We write $\gamma_1 \sim \gamma_2$ to express this relation. Then we know that, for functions

[1]The power of algebra often rests on using good notations !

satisfying the assumptions of the principle of monodromy (theorem 5.1.2), one has $f^{\gamma_1} = f^{\gamma_2}$:

$$\gamma_1 \sim \gamma_2 \Longrightarrow f^{\gamma_1} = f^{\gamma_2}.$$

As a consequence, f^{γ} only depends on the *homotopy class* $[\gamma] \in \Pi_1(\Omega; a, b)$ of γ. Therefore, we can define:

$$f^{[\gamma]} := f^{\gamma}.$$

Then the principle of conservation of algebraic and differential relations says that we have a map:

$$\Pi_1(\Omega; a, b) \to \mathrm{Iso}_{\mathbf{C}-algdiff}(\tilde{O}_a, \tilde{O}_b).$$

We have added something to the principle of conservation here. First, that analytic continuation sends $\tilde{O}_a$ to $\tilde{O}_b$; second, that it is bijective. Both statements come from the fact that analytic continuation can be reversed by going along the inverse path.

4. Remember from the course of topology that homotopy is compatible with the composition of pathes, so that for pathes γ_1 from a to b and γ_2 from b to c, and for their homotopy classes $[\gamma_1] \in \Pi_1(\Omega; a, b)$ and $[\gamma_2] \in \Pi_1(\Omega; b, c)$, one can define the product $[\gamma_1].[\gamma_2] \in \Pi_1(\Omega; a, c)$ in such a way that:

$$[\gamma_1].[\gamma_2] = [\gamma_1.\gamma_2].$$

Then, the previous relation on the effect of composition of pathes becomes:

$$f^{[\gamma_1.\gamma_2]} = (f^{[\gamma_1]})^{[\gamma_2]}.$$

5. Suppose now that $a = b$: the case of loop is the most interesting of all. Then $\Pi_1(\Omega; a, b)$ is $\pi_1(\Omega; a)$, the fundamental group of Ω with base point a, and we have a map:

$$\phi : \pi_1(\Omega; a) \to \mathrm{Aut}_{\mathbf{C}-algdiff}(\tilde{O}_a),$$

the group of automorphisms of the differential algebra $\tilde{O}_a$. Moreover, the equality $f^{[\gamma_1.\gamma_2]} = (f^{[\gamma_1]})^{[\gamma_2]}$ can be translated as: $\phi(xy) = \phi(y)\phi(x)$ (taking $x := [\gamma_1]$ and $y := [\gamma_2]$). Therefore, ϕ is an anti-morphism of groups.

The result can be summarized as follows:

Theorem 5.2.1 *The group $\pi_1(\Omega; a)$ operates at right on the differential algebra $\tilde{O}_a$.*

□

Remark 5.2.2 The fact that ϕ is an anti-morphism instead of a morphism is unavoidable if one wants to keep intuitive notations. Some books write $\gamma_2.\gamma_1$ what we have written $\gamma_1.\gamma_2$ and then they have a morphism. With their convention, the result of analytic continuation of f along γ is written $\gamma.f$ and one has: $[\gamma_2.\gamma_1].f = [\gamma_2]([\gamma_1].f)$, that is, the fundamental group operates at left, which is more usual. But I find awkward that notation for the composition of pathes.

Corollary 5.2.3 *The fixed set of the operation of $\pi_1(\Omega; a)$ on $\tilde{O}_a$ is $O(\Omega)$.*

Proof. - Indeed, if a germ can be continuated everywhere without ambiguity, it defines a *global* analytic function. □

.

Note how this result looks like a theorem from Galois theory !

Exercice 5.2.4 Take $\Omega := \mathbf{C}^*$ and $a := 1$, so that $\pi_1(\Omega; a)$ is isomorphic to $\mathbf{Z}$. Show that the linear space generated by 1 and log is stable under the operation of $\pi_1(\Omega; a)$ and describe the induced action.

5.3 A first look at differential equations with a singularity

We have already used, in section 1.5, the exponential of a matrix to solve the differential equation with constant coefficients $X' = AX$, where $A \in \mathrm{Mat}_n(\mathbf{C})$. We shall presntly use the logarithm function to solve the (very simple) *singular* equation $zX' = AX$. It is said to be singular because $X' = z^{-1}AX$ and $z^{-1}A$ is not defined at 0. We shall solve it on $\mathbf{C}^*$, that is, we shall look for an analytic solution $X : \mathbf{C}^* \to \mathbf{C}^n$.

Lemma 5.3.1 *Let $z_0 \in \mathbf{C}^*$ and let L be a determination of the logarithm in a domain Ω containing z_0. Then the matrix-valued function $\mathcal{X}(z) := e^{(L(z)-L(z_0))A}$ defined in Ω and with values in $\mathrm{Mat}_n(\mathbf{C})$ (actually in $GL_n(\mathbf{C})$) satisfies the equation:*

$$\mathcal{X}'(z) = (z^{-1}A)\mathcal{X}(z) = \mathcal{X}(z)(z^{-1}A).$$

Proof. - This is an immediate consequence of the following general fact. □

Lemma 5.3.2 *Let $M(z)$ be a matrix-valued analytic function on Ω be such that, for all $z \in \Omega$, $M(z)M'(z) = M'(z)M(z)$. Then e^M is analytic on Ω and $(e^M)' = e^M M' = M' e^M$.*

Proof. - Since $MM' = M'M$, Leibniz formula applied to M^k gives $(M^k)' = kM^{k-1}M' = M'(kM^{k-1})$. The rest of the proof is standard. □

Returning to our differential equation, we conclude again by a particular case of Cauchy theorem for complex analytic differential equations:

Theorem 5.3.3 *Let $Sol(z^{-1}A, \Omega) \subset O(\Omega)^n$ be the set of solutions of our differential equation in Ω. Then the map $X \mapsto X(z_0)$ from $Sol(z^{-1}A, \Omega)$ to $\mathbf{C}^n$ is an isomorphism of linear spaces.*

Proof. - It is clear that $Sol(z^{-1}A, \Omega)$ is a linear subspace of $O(\Omega)^n$ and that the map $X \mapsto X(z_0)$ is $\mathbf{C}$-linear.We are going to prove that its inverse is the map $X_0 \mapsto \mathcal{X}X_0$ from $\mathbf{C}^n$ to $O(\Omega)^n$.
Indeed, an immediate calculation shows that $\mathcal{X}X_0$ is a solution of the Cauchy problem $\begin{cases} X' = (z^{-1}A)X, \\ X(z_0) = X_0, \end{cases}$
and we have to see that it is the only one. But any solution X can be written $X = \mathcal{X}Y$ (since $\mathcal{X}(z)$ is invertible for every z) and the Leibniz rule, along with the first lemma and the equality $\mathcal{X}(z_0) = I_n$, then gives $Y' = 0$ and $Y(z_0) = X_0$. □

Note that this applies to all domains $\Omega \subset \mathbf{C}^*$ around z_0 on which there is a determination of the logarithm, in particular, to all simply connected domains. Therefore, all the corresponding spaces of solutions $Sol(z^{-1}A,\Omega)$ are isomorphic to each other.

Corollary 5.3.4 *The space $Sol(z^{-1}A)_{z_0} \subset \mathcal{O}_{z_0}^n$ of germs at z_0 of solutions of the differential equation is isomorphic to the space $\mathbf{C}^n$ of initial conditions, through the map $X \mapsto X(z_0)$.*

Example 5.3.5 To solve $\begin{cases} f' = 1/z \\ f(1) = 0 \end{cases}$, we solve the equivalent Cauchy problem $\begin{cases} zf'' + f' = 0 \\ f(1) = 0 \\ f'(1) = 1 \end{cases}$ (because $zf'' + f' = (zf')'$). We make it into a vectorial differential equation of order 1 by putting $X := \begin{pmatrix} f \\ zf' \end{pmatrix}$. Our problem then boils down to $\begin{cases} X' = (z^{-1}A)X \\ X(1) = X_0 \end{cases}$, where $A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $X_0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Here $A^2 = 0_2$, so that $e^{A\log z} = I_2 + A\log z = \begin{pmatrix} 1 & \log z \\ 0 & 1 \end{pmatrix}$, which gives in the end $f = \log z$ and $zf' = 1$: this is correct and consistent.

Monodromy of the solutions. With the same notations as above, let γ be a loop in $\mathbf{C}^*$ based at z_0. Then the result of the analytic continuation of $L(z)$ along γ is $L(z) + 2\mathrm{i}\pi k$, where $k := I(0,\gamma)$; the result of the analytic continuation of $\mathcal{X}(z)$ along γ is therefore $\mathcal{X}(z)e^{2\mathrm{i}\pi kA} = e^{2\mathrm{i}\pi kA}\mathcal{X}(z)$; and the result of the analytic continuation of a solution $X(z) = \mathcal{X}(z)X_0$ along γ is $\mathcal{X}(z)e^{2\mathrm{i}\pi kA}X_0$, that is, the solution with initial condition $e^{2\mathrm{i}\pi kA}X_0$. We express this fact by a *commutative diagram*:

$$\begin{tikzcd}
Sol(z^{-1}A)_{z_0} \arrow[r, "X \mapsto X(z_0)"] \arrow[d, "X \mapsto X^\gamma"'] & \mathbf{C}^n \arrow[d, "X_0 \mapsto e^{2\mathrm{i}\pi kA}X_0"] \\
Sol(z^{-1}A)_{z_0} \arrow[r, "X \mapsto X(z_0)"] & \mathbf{C}^n
\end{tikzcd}$$

Example 5.3.6 In the case of the last example, we get $e^{2\mathrm{i}\pi kA} = \begin{pmatrix} 1 & 2\mathrm{i}\pi k \\ 0 & 1 \end{pmatrix}$. Therefore, the solution $f = a + b\log z$ with initial conditions $f(1) = a$ and $(zf')(1) = b$ is transformed into the solution with initial conditions $\begin{pmatrix} 1 & 2\mathrm{i}\pi k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a + 2\mathrm{i}\pi kb \\ b \end{pmatrix}$, that is the function $f^\gamma = (a + 2\mathrm{i}\pi kb) + b\log z$: this is consistent.

Exercice 5.3.7 Solve in the same way $z^2 f'' + zf' + f = 0$ and find its monodromy.

Part II

The Riemann-Hilbert correspondence

Chapter 6

Two basic equations and their monodromy

Remark 6.0.8 The first section 6.1 was written to be the introduction to the whole course, before I added the preliminary part on complex analytic functions. Now, it is too long and there is a lot of redundancy with part I, but I did not have the courage to rewrite it: so the reader should browse it rather quickly. However, the following section 6.2 is *not* redundant and should be studied carefully.

6.1 The "characters" z^α

Let $\alpha \in \mathbf{Q}$ (the rational numbers) and assume that α is not a rational integer: $\alpha \notin \mathbf{Z}$. How can we define z^α for complex values $z \in \mathbf{C}$? If z is a strictly positive real, $z \in \mathbf{R}_+^*$, then one can use the real logarithm and complex exponential functions and put $z^r := \exp(\alpha \ln z)$. In the complex domain however, the logarithm function is not globally defined (as we saw in chapter 4), so we expect to obtain a "multivalued" function, that is local determinations and monodromy. In order to introduce the present chapter on complex differential equations, and also in order to define a family of useful basic functions for the sequel, we shall tackle the problem differently.

If $\alpha = p/q$ with $p \in \mathbf{Z}$ and $q \in \mathbf{N}^*$ (a non zero natural integer), we can assume that p and q are relatively prime; and, since $\alpha \notin \mathbf{Z}$, we know that $q \geq 2$. Then z^α must be a complex number $w \in \mathbf{C}$ such that $w^q = z^p$. If $z \neq 0$, writing $z = re^{\mathrm{i}\theta}$ with $r > 0$ and $\theta \in \mathbf{R}$, one finds that there are q such q^{th} roots of z^p; these are the complex numbers $r^\alpha e^{\mathrm{i}\alpha\theta} j$, where $r^\alpha := \exp(\alpha \ln r)$ (since $r \in \mathbf{R}_+^*$, this makes sense !) and where j is a q^{th} root of unity:

$$j \in \mu_q := \{e^{2k\mathrm{i}\pi/q} \mid 0 \leq k \leq q-1\}.$$

Note that r^α can also be defined in a more elementary way as $\sqrt[q]{r^p}$.

Thus, for each particular non zero complex number z, the fractional power z^α must be chosen among q possible values. In various senses, it is generally not possible to make such choices for all $z \in \mathbf{C}^*$ in a consistent way. Here is an example:

Exercice 6.1.1 (i) Show that putting $z^\alpha := \exp(\alpha \ln z)$ for $z > 0$, one has the rule: $\forall z_1, z_2 \in \mathbf{R}+^*$, $(z_1 z_2)^\alpha = (z^\alpha)(z_2^r)$.

(ii) Taking for example $\alpha := 1/2$, show that there is no way to define $\sqrt{z} = z^{1/2}$ for all $z \in \mathbf{C}^*$ in such a way that $(\sqrt{z})^2 = z$ and that moreover $\forall z_1, z_2 \in \mathbf{R}+^*$, $\sqrt{z_1 z_2} = \sqrt{z_1}\sqrt{z_2}$. (Hint: try to find a square root for -1.).

There is no global definition of z^α on $\mathbf{C}^*$. In this course we shall be more concerned with analytical aspects (although we intend to approach them through algebra) and we will rather insist on the impossibility to define z^α in such a way that it depends continuously on z:

Lemma 6.1.2 *We keep the same notations and assumptions about α, p, q. Let V be a punctured neighborhood of 0 in $\mathbf{C}$. Then there is no continuous function $f : V \to \mathbf{C}$ such that $\forall z \in V$, $(f(z))^q = z^p$.*

Proof. - By definition, V is a subset of $\mathbf{C}^*$ which contains some non trivial punctured disk centered at 0:

$$V \supset \overset{\circ}{\mathrm{D}}(0,t) \setminus \{0\}, \text{ where } t > 0.$$

Then, for some fixed $0 < s < t$, and for any $\theta \in \mathbf{R}$, one can define $g(\theta) := \dfrac{f(se^{\mathrm{i}\theta})}{s^\alpha e^{\mathrm{i}\alpha\theta}}\cdot$ This is a continuous function of θ and it satisfies:

$$\forall \theta \in \mathbf{R} \text{ , } \big(g(\theta)\big)^q = \frac{s^p e^{\mathrm{i}p\theta}}{s^p e^{\mathrm{i}p\theta}} = 1.$$

Therefore, the continuous function g maps the connected set $\mathbf{R}$ to the discrete finite set μ_q, therefore it must be constant: there exists a fixed $j \in \mu_q$ such that:

$$\forall \theta \in \mathbf{R} \text{ , } f(se^{\mathrm{i}\theta}) = js^\alpha e^{\mathrm{i}\alpha\theta}.$$

Now, replacing θ by $\theta + 2\pi$, the left hand side does not change, while the right hand side (which is $\neq 0$) is multiplied by $e^{2\mathrm{i}\pi\alpha}$. This imples that $e^{2\mathrm{i}\pi\alpha} = 1$, which contradicts the assumption that $\alpha \notin \mathbf{Z}$. $\Box$

The lemma obviously implies that there is no way to define z^α as a continuous map on $\mathbf{C}^*$. What we are going to do is to look for local definitions, that is, continuous functions on sufficiently small neighborhoods of all non zero complex numbers. Then we shall consider the possibility of patching together these local objects.

Exercice 6.1.3 Use the calculation of the monodromy of $\sqrt{z}$ in example 5.1.4, page 32, to give another proof that there is no global determination of $\sqrt{z}$.

Transforming an algebraic equation into a differential one. We fix $z_0 \in \mathbf{C}^*$ and choose a particular $w_0 \in \mathbf{C}^*$ such that $w_0^q = z_0^p$. This w_0 will serve as a kind of "initial condition" to define the function $z \mapsto z^\alpha$ in the neighborhood of z_0.

Proposition 6.1.4 *(i) There exists a unique power series $f(z) = \sum a_n (z - z_0)^n$ such that $f(z_0) = w_0$ and $\big(f(z)\big)^q = z^p$. Its radius of convergence is $|z_0|$.*

(ii) Any function g defined and continuous in a connected neighborhood $V \subset \overset{\circ}{D}(z_0, |z_0|)$ of z_0 and such that $\big(g(z)\big)^q = z^p$ in V is a constant multiple of f.

Proof. - (i) We first note that, for any power series f defined in a connected neighborhood U of z_0, one has the equivalence:

$$\begin{cases} f(z_0) = w_0, \\ \forall z \in U \ , \ \big(f(z)\big)^q = z^p \end{cases} \iff \begin{cases} f(z_0) = w_0, \\ \forall z \in U \ , \ zf'(z) = \alpha f(z). \end{cases}$$

Indeed, if the algebraic equation $\big(f(z)\big)^q = z^p$ is true, then f does not vanish anywhere on U and one can take the logarithmic derivatives on both sides, which indeeds yields the differential equation $zf'(z) = \alpha f(z)$. Conversely, the differential equation implies that the function $\big(f(z)\big)^q/z^p$ has a trivial derivative, hence (U being connected) it is constant; the initial condition then implies that it is equal to 1. Now the first assertion follows from the similar one for differential equations, which will be proved further below (theorem 6.1.8).
(ii) If g is such a solution, then $(g/f)^q = z^p/z^p = 1$ on V. The continuous map g/f sends the connected set V to the discrete set μ_q, therefore it is constant. □

From now on, we shall therefore study the differential equation $zf' = \alpha f$, where $\alpha \in \mathbf{C}$ is an arbitrary complex number. Indeed there is no reason to restrict to rational α. We first prove again the impossibility of a global solution except in trivial cases.

Lemma 6.1.5 *If $\alpha \notin \mathbf{Z}$, the differential equation $zf' = \alpha f$ has no non trivial solution in any punctured neighborhood of* 0.

Proof. - Of course, if $\alpha \in \mathbf{Z}$, the solution z^α is well defined in $\mathbf{C}$ or $\mathbf{C}^*$ according to the sign of α; and for arbitrary α, there is always the trivial solution $f = 0$.
Suppose that f is a non trivial solution in some punctured disk $\overset{\circ}{\mathrm{D}}(0,t) \setminus \{0\}$, where $t > 0$. Then, for some fixed $0 < s < t$, and for any $\theta \in \mathbf{R}$, one can define $g(\theta) := \dfrac{f(se^{\mathrm{i}\theta})}{e^{\mathrm{i}\alpha\theta}} \cdot$ This is a differentiable function of θ and a simple computation shows that it satisfies:

$$\forall \theta \in \mathbf{R} \ , g'(\theta) = 0.$$

Therefore, g is constant and there exists c such that:

$$\forall \theta \in \mathbf{R} \ , \ f(se^{\mathrm{i}\theta}) = ce^{\mathrm{i}\alpha\theta}.$$

Now, replacing θ by $\theta + 2\pi$, the left hand side does not change, while the right hand side (which is $\neq 0$) is multiplied by $e^{2\mathrm{i}\pi\alpha}$. This implies that $e^{2\mathrm{i}\pi\alpha} = 1$, which contradicts the assumption that $\alpha \notin \mathbf{Z}$. □

Exercice 6.1.6 Give another proof that $\alpha \in \mathbf{Z}$ by integrating $f'/f = \alpha/z$ on a small circle centered at 0 and by using Cauchy residue formula.

Local solutions of the differential equation. Before stating the theorem, let us make some preliminary remarks about the solutions of the differential equation $zf' = \alpha f$.

1. On $\mathbf{R}_+^* =]0, +\infty[$, there is the obvious solution $z^\alpha := \exp(\alpha \ln z)$. Moreover, any solution defined on a connected open set of $\mathbf{R}_+^*$ (*i.e.* on an interval) has to be a constant multiple of this one, because the differential equation implies that f/z^α has zero derivative.

2. If V is any open subset of $\mathbf{C}^*$, we shall write $\mathcal{F}(V)$ the set of solutions of the differential equation $zf' = \alpha f$. Then $\mathcal{F}(V)$ is a linear space over the field $\mathbf{C}$ of complex numbers. Indeed, if f_1, f_2 are any solutions and if λ_1, λ_2 are any complex coefficients, it is obvious that $\lambda_1 f_1 + \lambda_2 f_2$ is a solution.

3. Suppose moreover that the open set $V \subset \mathbf{C}^*$ is connected and suppose that f_0 is a non trivial solution on V. Then, for any solution f, the meromorphic function $g := f/f_0$ has a trivial derivative (its logarithmic derivative is $g'/g = f'/f - f_0'/f_0 = \alpha/z - \alpha/z = 0$) therefore g is constant on V. This means that all solutions f are constant multiples of f_0.

4. As a corollary, if V is connected, there is a dichotomy:

   - Either $\mathcal{F}(V) = \{0\}$, there is no non trivial solution on V. This happens for instance if $V = \mathbf{C}^*$, more generally if V is a punctured neighborood of 0 (under the assumption that $\alpha \notin \mathbf{Z}$) and even more generally if it contains the image of a loop γ such that $I(0, \gamma) \neq 0$ (see the last exercice[1]).
   - Or $\mathcal{F}(V)$ is generated by any of its non zero elements, that is, it has dimension 1.

Remark 6.1.7 When V is empty, we shall take the convention that $\mathcal{F}(V) = \{0\}$, the trivial linear space. The reader can check that all our general assertions shall remain true in that degenerate case.

Theorem 6.1.8 *For any $z_0 \in \mathbf{C}^*$ and $w_0 \in \mathbf{C}$, the differential equation (with initial condition):*

(6.1.8.1)
$$\begin{cases} f(z_0) = w_0, \\ zf' = \alpha f, \end{cases}$$

has a unique power series solution $f = \sum_{n\geq 0} a_n (z - z_0)^n$. If $w_0 \neq 0$ and $\alpha \notin \mathbf{N}$, the radius of convergence of f is exactly $|z_0|$.

Proof. - The initial condition $f(z_0) = w_0$ translates to $a_0 = w_0$, which determines the first coefficient. Then, from the calculation:

$$zf' = (z - z_0)f' + z_0 f' = \sum_{n\geq 0} n a_n (z - z_0)^n + z_0 \sum_{n\geq 0} (n+1) a_{n+1} (z - z_0)^n = \sum_{n\geq 0} (n a_n + (n+1) a_{n+1} z_0)(z - z_0)^n,$$

one gets the recursive relation:

$$\forall n \in \mathbf{N}\,,\; n a_n + (n+1) a_{n+1} z_0 = \alpha a_n \Longrightarrow \forall n \in \mathbf{N}\,,\; (n+1) a_{n+1} z_0 = (\alpha - n) a_n \Longrightarrow \forall n \in \mathbf{N}\,,\; a_{n+1} = z_0^{-1} \frac{\alpha - n}{n+1} a_n.$$

Solving this, we get:

$$\forall n \in \mathbf{N}\,,\; a_n = w_0 \binom{\alpha}{n} z_0^{-n},$$

[1]The exercice requires a path of class $\mathcal{C}^1$, but one can prove that any continuous path with values in an open set of $\mathbf{C}^*$ is homotopic to such a path. See for instance the article by R. Vyborny, "On the use of a differentiable homotopy in the proof of the Cauchy theorem", in the American Matematical Monthly, 1979; or the book by Madsen and Tornehave, "From Calculus to Cohomology".

where the generalized binomial coefficients $\binom{\alpha}{n}$ have been defined in chapter 2. Thus:

$$f(z) = w_0 \sum_{n\geq 0} \binom{\alpha}{n} \left(\frac{z-z_0}{z_0}\right)^n. \tag{6.1.8.2}$$

One also recognizes the generalized Newton binomial formula, already met as a formal power series:

$$(1+u)^\alpha := \sum_{n\geq 0} \binom{\alpha}{n} u^n. \tag{6.1.8.3}$$

(As we are going to see, this is only well defined for $|u| < 1$.) Then our solution can be expressed as:

$$f(z) = w_0 \left(1 + \frac{z-z_0}{z_0}\right)^\alpha.$$

Of course, if $w_0 = 0$ this is trivial, so assume that $w_0 \neq 0$. Then, as we saw, if $\alpha \in \mathbf{N}$, this is a polynomial, so that the radius of convergence is infinite, so assume that $\alpha \notin \mathbf{N}$. Then the coefficients a_n are all non zero, and:

$$\left| \frac{\binom{\alpha}{n+1}}{\binom{\alpha}{n}} \right| = \left| \frac{\alpha - n}{n+1} \right| \underset{n\to+\infty}{\longrightarrow} 1,$$

so that the radius of convergence of the power series (6.1.8.3) is 1 and the radius of convergence of the power series (6.1.8.2) is $|z_0|$. □

Exercice 6.1.9 (i) Remember that the series (6.1.8.3) is the unique power series solution of the differential equation:

$$\begin{cases} f(0) = 1, \\ (1+u)f' = \alpha f. \end{cases}$$

Using only this characterization, prove the following formula:

$$(1+u)^{\alpha+\beta} = (1+u)^\alpha (1+u)^\beta.$$

(ii) As an application, prove the following formula:

$$\forall n \in \mathbf{N}\,,\ \sum_{i+j=n} \binom{\alpha}{i}\binom{\beta}{j} = \binom{\alpha+\beta}{n}.$$

Exercice 6.1.10 Prove the following formulas:

$$\forall p \in \mathbf{N}^*\,,\ \frac{1}{(1+u)^p} = \sum_{n\geq 0} (-1)^n \frac{(p+n-1)!}{(p-1)!n!} u^n,$$

$$\sqrt{1+u} = 1 + \frac{1}{2} \sum_{n\geq 1} \left(\frac{-1}{4}\right)^{n-1} \frac{1}{n} \binom{2n-2}{n-1} u^n = 1 + \frac{u}{2} - \frac{u^2}{8} + \cdots,$$

$$\frac{1}{\sqrt{1+u}} = \sum_{n\geq 0} \left(\frac{-1}{4}\right)^n \binom{2n}{n} u^n = 1 - \frac{u}{2} + \frac{3u^2}{8} + \cdots$$

The principal determination. Now consider all the open disks $\overset{\circ}{\mathrm{D}}(z_0, |z_0|)$ with $z_0 > 0$. This is an increasing family of open subsets of the complex plane $\mathbf{C}$, with union the right ("eastern") half plane:

$$\bigcup_{z_0>0} \overset{\circ}{\mathrm{D}}(z_0, |z_0|) = H_0 := \{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}.$$

The elements of H_0 are the complex numbers which can be written $z = re^{\mathrm{i}\theta}$ with $r > 0$ and $-\pi/2 < \theta < \pi/2$.

For any disk $\overset{\circ}{\mathrm{D}}(z_0, |z_0|)$ with $z_0 > 0$, there is a unique solution of the differential equation (6.1.8.1) with initial condition $f(z_0) = z_0^\alpha := e^{\alpha \ln z_0}$; we temporarily write $f_{z_0} \in \mathcal{F}\left(\overset{\circ}{\mathrm{D}}(z_0, |z_0|)\right)$ for this solution.

Lemma 6.1.11 *If $0 < z_0 < z_1$, then the restriction of f_{z_1} to $\overset{\circ}{D}(z_0, |z_0|) \subset \overset{\circ}{D}(z_1, |z_1|)$ is equal to f_{z_0}. Temporarily write f the unique function on H_0 which, for all $z_0 > 0$, restricts to f_{z_0} on $\overset{\circ}{D}(z_0, |z_0|)$ (this exists by the previous assertion). Then the restriction of f to $\mathbf{R}_+^*$ is the function $z \mapsto z^\alpha := e^{\alpha \ln z}$.*

Proof. - The restriction of f_{z_0} to $\overset{\circ}{\mathrm{D}}(z_0, |z_0|) \cap \mathbf{R}_+^* =]0, 2z_0[$ is equal to the function $z \mapsto z^\alpha$: this is because they satisfy the same differential equation with the same initial condition on that open interval. Therefore, if $0 < z_0 < z_1$, the functions f_{z_1} and f_{z_0} have the same value at z_0, hence they are equal on $\overset{\circ}{\mathrm{D}}(z_0, |z_0|)$ by the unicity property in theorem 6.1.8. □

From now on, we shall write $z^\alpha := f(z)$. This function is the unique $f \in \mathcal{F}(H_0)$ such that $f(1) = 1$.

Proposition 6.1.12 *Let $z \in H_0$ be written $z = re^{i\theta}$ with $r > 0$ and $-\pi/2 < \theta < \pi/2$. Then $z^\alpha = r^\alpha e^{i\alpha\theta}$.*

Proof. - Let $g(re^{\mathrm{i}\theta}) := r^\alpha e^{\mathrm{i}\alpha\theta} = e^{\alpha(\ln r + \mathrm{i}\theta)}$. This defines a function from H_0 to $\mathbf{C}^*$ which coincides with z^α on $\mathbf{R}_+^*$. Differentiating the relations $r^2 = x^2 + y^2$ and $\tan\theta = y/x$, one draws first:

$$r\,dr = x\,dx + y\,dy \text{ and } (1 + \tan^2\theta)\,d\theta = \frac{x\,dy - y\,dx}{x^2},$$

whence:

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r},$$
$$\frac{\partial \theta}{\partial x} = \frac{-y}{r^2}, \frac{\partial \theta}{\partial y} = \frac{x}{r^2},$$

from which one computes:

$$\frac{\partial g}{\partial x} = \alpha \frac{x - \mathrm{i}y}{r^2} g,$$
$$\frac{\partial g}{\partial y} = \alpha \frac{y + \mathrm{i}x}{r^2} g.$$

Thus, $\dfrac{\partial g}{\partial y} = \mathrm{i}\dfrac{\partial g}{\partial x}$, which proves that g is holomorphic on $\mathbf{C}^*$. Moreover, $zg' = \alpha g$, which shows that $g \in \mathcal{F}(H_0)$. Since $g(1) = 1$, the function g is equal to the function $z \mapsto z^\alpha$. $\square$

Remark 6.1.13 If $\alpha \in \mathbf{Z}$, the above formula is trivial. However, if $\alpha \notin \mathbf{Z}$ the formula can only be applied if $-\pi/2 < \theta < \pi/2$. For instance, trying to compute 1^α using the representation $1 = 1.e^{2\mathrm{i}\pi}$ would yield the incorrect result $e^{2\mathrm{i}\pi\alpha} \neq 1$.

Corollary 6.1.14 *The unique function $f \in \mathcal{F}\left(\overset{\circ}{D}(z_0, |z_0|)\right)$ such that $f(z_0) = w_0$ is defined by $z \mapsto w_0(z/z_0)^\alpha = w_0(r/r_0)^\alpha e^{\mathrm{i}\alpha(\theta-\theta_0)}$, where $z = re^{i\theta}$ and $z_0 = r_0e^{i\theta_0}$, with $r, r_0 > 0$ and $-\pi/2 < \theta - \theta_0 < \pi/2$.*

Corollary 6.1.15 *Let $H_{\theta_0} := e^{i\theta_0}H_0 = \{z \in \mathbf{C} \mid Re(z/e^{i\theta_0}) > 0\}$. Then the function defined by:*

$$re^{i\theta} \mapsto r^\alpha e^{i\alpha(\theta-\theta_0)}, \text{ where } r > 0 \text{ and } -\pi/2 < \theta - \theta_0 < \pi/2$$

generates $\mathcal{F}(H_{\theta_0})$.

How can one patch the local solutions. We shall now try to "patch together" these "local" solutions. We start with a simple example.

Example 6.1.16 We consider the upper half-plane (also called *Poincaré half-plane*) $H_{\pi/2} = \mathrm{i}H_0 = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$. Then $H_0 \cap H_{\pi/2}$ is the upper right quadrant. The restriction of $z^\alpha \in \mathcal{F}(H_0)$ to $H_0 \cap H_{\pi/2}$ has a unique extension to $H_{\pi/2}$. This extension can be described as follows: one chooses an arbitrary $z_0 \in H_0 \cap H_{\pi/2}$; then this extension is the unique $f \in \mathcal{F}(H_{\pi/2})$ satisfying the initial condition $f(z_0) = z_0^\alpha$. Note that this makes sense as an initial condition for an element of $\mathcal{F}(H_{\pi/2})$, since $z_0 \in H_{\pi/2}$; and the right hand hand side z_0^α makes sense since $z_0 \in H_0$.
Also note that the particular choice of z_0 does not matter. Indeed, in any case, the unique $f \in \mathcal{F}(H_{\pi/2})$ such that $f(z_0) = z_0^\alpha$ will have to coincide with z^α on $H_0 \cap H_{\pi/2}$. (The reader should check this statement!)
To compute the function f explicitly, we first choose a nice particular value of z_0. We shall take $z_0 := \dfrac{1+\mathrm{i}}{\sqrt{2}} = e^{\mathrm{i}\pi/4}$. Since $-\pi/2 < \pi/4 < \pi/2$, using proposition 6.1.12, we see that $z_0^\alpha = e^{\mathrm{i}\alpha\pi/4}$.
Now, by corollary 6.1.15, the function f on $H_{\pi/2}$ can be computed as $f(z) = w_0(z/\mathrm{i})^\alpha$, which makes sense because $z \in H_{\pi/2} \Rightarrow z/\mathrm{i} \in H_0$. We determine w_0 using the initial condition at z_0. Since $z_0/\mathrm{i} = e^{\mathrm{i}\pi/4 - \mathrm{i}\pi/2} = e^{-\mathrm{i}\pi/4}$ and since $-\pi/2 < -\pi/4 < \pi/2$, using proposition 6.1.12, we see that $(z_0/\mathrm{i})^\alpha = e^{-\mathrm{i}\alpha\pi/4}$. Now, from the initial condition $f(z_0) = z_0^\alpha = e^{\mathrm{i}\alpha\pi/4} = w_0e^{-\mathrm{i}\alpha\pi/4}$, we conclude that $w_0 = e^{\mathrm{i}\alpha\pi/2}$ and we obtain the formula for f:

$$\forall z \in H_{\pi/2}\,,\ f(z) = e^{\mathrm{i}\alpha\pi/2}(z/\mathrm{i})^\alpha.$$

In some sense, we have tried to continuate the function z^α along a path that starts at $z_0 := 1$ and that turns around the origin 0 in the positive direction (up, then left). We can go further by considering any half-plane H_θ which meets H_0, that is, any half-plane except $-H_0 = H_\pi = H_{-\pi}$. So we take any θ_0 such that $-\pi < \theta_0 < \pi$. Then $H_0 \cap H_{\theta_0}$ is connected, it is actually a sector:

$$H_0 \cap H_{\theta_0} = \begin{cases} \{re^{\mathrm{i}\theta} \mid r > 0 \text{ and } \theta_0 - \pi/2 < \theta < \pi/2\} \text{ if } \theta_0 > 0, \\ \{re^{\mathrm{i}\theta} \mid r > 0 \text{ and } -\pi/2 < \theta < \theta_0 + \pi/2\} \text{ if } \theta_0 < 0. \end{cases}$$

Lemma 6.1.17 *Under these conditions, there is a unique function $f \in \mathcal{F}(H_{\theta_0})$ which coincides with z^α on their common domain of definition. This function is given by the formula:*

$$\forall \theta \in H_{\theta_0}\ ,\ f(z) = e^{i\alpha\theta_0}(z/e^{i\theta})^\alpha.$$

Proof. - Putting $f(z) = C(z/e^{i\theta})^\alpha$ with $C \in \mathbf{C}^*$, we just have to replace z by a particular point of $H_0 \cap H_{\theta_0}$ to be able to determine the unknown factor C. We choose $z := e^{i\theta_0/2}$ and compute:

$$z^\alpha = e^{i\alpha\theta_0/2} \text{ since } -\pi/2 < \theta_0/2 < \pi/2,$$
$$(z/e^{i\theta})^\alpha = e^{-i\alpha\theta_0/2} \text{ since } -\pi/2 < -\theta_0/2 < \pi/2,$$

and we conclude that $C = e^{i\alpha\theta_0}$. □

So suppose we now want to compute $(-1)^\alpha$. To begin with, this is not really defined since $-1 \notin H_0$. So a natural way to proceed is to choose some θ_0 as before, to take the unique continuation of z^α into a function $f \in H_{\theta_0}$ and to evaluate $f(-1)$. As soon as $H_{\theta_0} \neq H_0$ one indeed has $-1 \in H_{\theta_0}$, so this is guaranteed to work. We shall try two distinct possibilities for θ_0.

Example 6.1.18 Take $\theta_0 \in]\pi/2, \pi[$. Then $-1 = e^{i\pi}$ with $\theta_0 - \pi/2 < \pi < \theta_0 + \pi/2$, so that, from the lemma: $f(-1) = e^{\alpha i\pi}$.

Example 6.1.19 Take $\theta_0 \in]-\pi, -\pi/2[$. Then $-1 = e^{-i\pi}$ with $\theta_0 - \pi/2 < -\pi < \theta_0 + \pi/2$, so that, from the lemma: $f(-1) = e^{-\alpha i\pi}$.

Therefore, we have two candidate values for $(-1)^\alpha$, that is $e^{\alpha i\pi}$ and $e^{-\alpha i\pi}$. If $\alpha \notin \mathbf{Z}$, these values are distinct. Different continuations of z^α from a neighborhood of 1 to a neighborhood of -1 have given different results. This can be seen as an "explanation" of the impossibility of defining globally z^α on $\mathbf{C}^*$.

Analytic continuation and differential equations. Given that we are not in general able to continuate z^α to a solution of $zf' = \alpha f$ in the whole of $\mathbf{C}^*$, we might decide to relax our condition and look for functions satisfying weaker conditions. For instance, it is not too difficult to prove that the function z^α on H_0 can be extended (in many ways) to a function of which is indefinitely differentiable on C^* in its variables x, y. (Because of the wild behaviour of z^α near 0, we cannot hope for an extension to the whole of $\mathbf{C}$.)

The problem as we study it appeared in the XIXth century, when mathematicians like Gauss, Cauchy, Riemann … had discovered the marvelous properties of *analytic* functions of a complex variables. Now, such functions satisfy very strong "rigidity" properties. For instance, if a function f defined on a connected open set $U \subset \mathbf{C}$ satisfies an algebraic equation (like $f^q = z^p$) or a differential equation (like $zf' = \alpha f$), then all its analytic continuations satisfy the same equation. We shall now explain this sentence and prove it in a simple case; the general case for linear differential equations[2] will be tackled in the following chapters.

[2]The case of algebraic equations is studied in all books about "algebraic functions", or in the corresponding chapter of many books on complex functions, like the book of Ahlfors.

So assume either that $f^q = z^p$ or that $zf' = \alpha f$. Now consider an open connected set V such that $U \cap V \neq \emptyset$. Suppose there is an analytic function g on V such that f and g coincide on $U \cap V$. This implies that the function $h := g^q - z^p$ (in the first case) or $h := zg' - \alpha g$ (in the second case) has a trivial restriction to $U \cap V$. Since h is analytic by the *principle of analytic continuation*, this implies that h is trivial on V, thus that g is also a solution of the algebraic or the differential equation. The function g is called a *direct analytic continuation* of f. If we consider a direct analytic continuation of g, etc, we obtain various analytic continuations of f. All these continuations are solutions of the same algebraic or differential equation as f.

Riemann proved in two celebrated works[3] that algebraic functions and solutions of differential equations could be better understood through the properties of their analytic continuations (and also through the study of their singularities). Galois guessed that the ambiguities caused by the multiplicity of analytic continuations could be used to build a "Galois theory of transcendental functions" as he had done for algebraic equations satisfied by numbers.

A formal look at what has been done. We now try to understand the process of analytic continuation of the function z^α but more globally, for all functions $f \in \mathcal{F}(H_0)$ simultaneously. We recall that $\mathcal{F}(H_0)$ is a complex linear space and that it is generated by any of its non trivial elements, for instance by z^α: so the dimension of this complex space is 1.

If we choose any $\theta_0 \neq 0$ such that $-\pi < \theta_0 < \pi$, then $H_0 \cap H_{\theta_0}$ is non empty and connected. Restricting $f \in \mathcal{F}(H_0)$ to $H_0 \cap H_{\theta_0}$ gives a function g on $H_0 \cap H_{\theta_0}$ such that $zg' = \alpha g$, that is, an element $g \in \mathcal{F}(H_0 \cap H_{\theta_0})$. In this way, we obtain a map:

$$\begin{cases} \mathcal{F}(H_0) \to \mathcal{F}(H_0 \cap H_{\theta_0}), \\ f \mapsto f_{|H_0 \cap H_{\theta_0}}. \end{cases}$$

This map is obviously linear, and it follows from our previous arguments that it is bijective: it is an isomorphism.

In the same way, one defines an isomorphism:

$$\begin{cases} \mathcal{F}(H_{\theta_0}) \to \mathcal{F}(H_0 \cap H_{\theta_0}), \\ f \mapsto f_{|H_0 \cap H_{\theta_0}}. \end{cases}$$

Then by composition of the first isomorphism with the inverse of the second isomorphism:

$$\begin{cases} \mathcal{F}(H_0) \to \mathcal{F}(H_{\theta_0}), \\ f \mapsto \text{ the unique extension to } H_{\theta_0} \text{ of } f_{|H_0 \cap H_{\theta_0}}. \end{cases}$$

For instance, we proved that the image of z^α under this isomorphism is the function $e^{i\alpha\theta_0}(z/e^{i\theta})^\alpha$.

[3] I only know the french references: "Principes fondamentaux pour une théorie générale des fonctions d'une grandeur variable complexe," and "Contribution à la théorie des fonctions représentables par la série de Gauss $F(\alpha, \beta, \gamma; x)$", in "Oeuvres mathématiques"; the second one is the most relevant here.

Now we can prove in the same way that there are isomorphisms:

$$\begin{cases} \mathcal{F}(H_{\theta_0}) \to \mathcal{F}(H_\pi \cap H_{\theta_0}), \\ f \mapsto f_{|H_\pi \cap H_{\theta_0}}. \end{cases}$$

$$\begin{cases} \mathcal{F}(H_\pi) \to \mathcal{F}(H_\pi \cap H_{\theta_0}), \\ f \mapsto f_{|H_\pi \cap H_{\theta_0}}. \end{cases}$$

Then by composition of the first isomorphism with the inverse of the second isomorphism:

$$\begin{cases} \mathcal{F}(H_{\theta_0}) \to \mathcal{F}(H_\pi), \\ f \mapsto \text{ the unique extension to } H_\pi \text{ of } f_{|H_\pi \cap H_{\theta_0}}. \end{cases}$$

To summarize, we obtain an isomorphism $\mathcal{F}(H_0) \to \mathcal{F}(H_\pi)$ by following a complicated path:

$$\mathcal{F}(H_0) \to \mathcal{F}(H_0 \cap H_{\theta_0}) \leftarrow \mathcal{F}(H_{\theta_0}) \to \mathcal{F}(H_\pi \cap H_{\theta_0}) \leftarrow \mathcal{F}(H_\pi).$$

All the arrows are restriction maps. The path always goes from left to right, so we follow some arrows backwards! (Of course, this is possible because they are isomorphisma.)

Now, we shall try to characterize the isomorphism $\mathcal{F}(H_0) \to \mathcal{F}(H_\pi)$ thus defined. It is sufficient to compute the image f of the generator z^α of $\mathcal{F}(H_0)$. As an element of $\mathcal{F}(H_\pi)$, the function f must have the form $f(z) = C(z/e^{i\pi})^\alpha = C(-z)^\alpha$ (remember that here we have $z \in H_\pi$, so that $-z \in H_0$). The constant C can be determined by taking $z = -1$: we have $C = f(-1)$. This value has been computed in examples 6.1.18 and 6.1.19. There, we found that it depends on the choice of the intermediate plane:

$$\forall z \in H_\pi \,,\; f(z) = C(-z)^\alpha, \text{ where } C = \begin{cases} C_{up} := e^{\alpha i\pi} \text{ if } \theta_0 \in \left]\pi/2, \pi\right[, \\ C_{down} := e^{-\alpha i\pi} \text{ if } \theta_0 \in \left]-\pi, -\pi/2\right[. \end{cases}$$

Now we do some very simple linear algebra. We call u the generator of $\mathcal{F}(H_0)$ that we wrote until now z^α. We call v the generator of $\mathcal{F}(H_\pi)$ that we wrote until now $(-z)^\alpha$. We have defined two isomorphisms ϕ_{up} and ϕ_{down} from $\mathcal{F}(H_0)$ to $\mathcal{F}(H_\pi)$: one is such that $\phi_{up}(u) = C_{up}v$, the other is such that $\phi_{down}(u) = C_{down}v$. We can compose one of this isomorphisms with the inverse of the other and obtain an automorphism ψ of $\mathcal{F}(H_0)$, defined by:

$$\psi := \phi_{down}^{-1} \circ \phi_{up} : H_0 \to H_0.$$

This is characterized by the fact that $\psi(u) = \dfrac{C_{up}}{C_{down}} u = e^{2\alpha i\pi} u$. Since u is a generator of $\mathcal{F}(H_0)$, this implies:

$$\forall f \in \mathcal{F}(H_0) \,,\; \psi(f) = e^{2\alpha i\pi} f.$$

How can we understand the automorphism ψ of $\mathcal{F}(H_0)$? It has been obtained by following an even longer complicated path:

$$\mathcal{F}(H_0) \to \mathcal{F}(H_0 \cap H_{\theta'_0}) \leftarrow \mathcal{F}(H_{\theta'_0}) \to \mathcal{F}(H_\pi \cap H_{\theta'_0}) \leftarrow \mathcal{F}(H_\pi) \to \cdots$$

$$\cdots \to \mathcal{F}(H_\pi \cap H_{\theta''_0}) \leftarrow \mathcal{F}(H_{\theta''_0}) \to \mathcal{F}(H_0 \cap H_{\theta''_0}) \leftarrow \mathcal{F}(H_0).$$

(As before, all the arrows are restriction maps and we follow some arrows backwards.) Here we have taken two distinct intermediate halph-planes, one "up", characterized by $\theta'_0 \in]\pi/2,\pi[$, the other "down", characterized by $\theta''_0 \in]-\pi,-\pi/2[$. In some sense, we have turned around 0 in the positive sense (counterclockwise) and $f \in \mathcal{F}(H_0)$ has been changed in the process.

Exercice 6.1.20 Call D_θ the open disk $\overset{\circ}{\mathrm{D}}(e^{i\theta},1)$. Define isomorphisms:

$$\mathcal{F}(D_0) \to \mathcal{F}(D_0 \cap D_{\pi/2}) \leftarrow \mathcal{F}(D_{\pi/2}) \to \mathcal{F}(D_\pi \cap D_{\pi/2}) \leftarrow \mathcal{F}(D_\pi) \to \cdots$$

$$\cdots \to \mathcal{F}(D_\pi \cap D_{3\pi/2}) \leftarrow \mathcal{F}(D_{3\pi/2}) \to \mathcal{F}(D_{2\pi} \cap D_{3\pi/2}) \leftarrow \mathcal{F}(D_{2\pi}).$$

From the fact that $D_{2\pi} = D_0$, deduce an automorphism of $\mathcal{F}(D_0)$ and describe explicitly that automorphism.

6.2 A new look at the complex logarithm

Instead of the equation $f' = 1/z$, we shall here use its consequence:

$$zf'' + f' = 0, \tag{6.2.0.1}$$

because it is a linear homogeneous scalar differential equation, and so its solutions form a linear space. Of course, we have greatly increased the set of solutions beyond the logarithm; formally:

$$(zf')' = zf'' + f' = 0 \Longrightarrow zf' = a \Longrightarrow f = a\log z + b,$$

and we can only fix the two constants of integration by specifying initial conditions.

We saw that putting $X := \begin{pmatrix} f \\ zf' \end{pmatrix}$, we obtain an equivalent vectorial differential equation (or system):

$$X' = z^{-1}AX, \text{ where } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{6.2.0.2}$$

We shall now formalize more precisely in what sense they are equivalent. For any open set $U \subset \mathbf{C}$, let us write:

$\mathcal{F}_1(U) := \{f \in O(U) \mid f \text{ is a solution of } (6.2.0.1)\}$ and $\mathcal{F}_2(U) := \{X \in O(U)^2 \mid X \text{ is a solution of } (6.2.0.2)\}$.

These are two linear spaces, and the map $\phi : f \mapsto \begin{pmatrix} f \\ zf' \end{pmatrix}$ is an isomorphism from the first to the second. If we want to take in account initial conditions, we introduce the map $IC_2 : X \mapsto X(z_0)$ from $\mathcal{F}_2(U)$ to $\mathbf{C}^2$ and, for compatibility, a map IC_1 from $\mathcal{F}_1(U)$ to $\mathbf{C}^2$ defined as $f \mapsto (f(z_0), z_0f'(z_0))$. Then, we have a commutative diagram:

$$\begin{tikzcd} \mathcal{F}_1(U) \arrow[rr, "\phi"] \arrow[dr, "IC_1"'] & & \mathcal{F}_2(U) \arrow[dl, "IC_2"] \\ & \mathbf{C}^2 & \end{tikzcd}$$

In the course, we shall generally study the vectorial form (after having shown that it is equivalent to the scalar form), but in this particular case we shall rather use the scalar form (6.2.0.1). So, we simply write $\mathcal{F}(U)$ for $\mathcal{F}_1(U)$ (and IC for the initial map IC_1). We shall first study some particular cases, depending on the open set U. Then we shall see how all these spaces $\mathcal{F}(U)$ are globally related.

The singularity at 0. This is considered as a singularity because, if we write the equation in the form $f'' + p(z)f' + q(z)f$, the functions p and q are singular at 0 and the Cauchy theorem on differential equation (which we shall meet later) cannot be applied. Specifically, if we try to solve (6.2.0.1) with a power series $f := \sum a_n z^n$, we see that it is equivalent to:

$$zf'' + f' = \sum (n+1)^2 a_{n+1} z^n = 0 \Longleftrightarrow \forall n\ ,\ (n+1)^2 a_{n+1} = 0 \Longleftrightarrow \forall n \neq 0\ ,\ a_n = 0.$$

Therefore, the only solutions are constants, even on very small neighborhoods of 0. Note that we said nothing about the values of n: the argument applies as well to Laurent series and even generalized Laurent series, that is solutions which are analytic in a punctured neighborhood of 0. This is clearly not satisfying, since for an equation of order 2 we hope to obtain two arbitrary constants of integration. So, from now on, we shall only consider open sets $U \subset \mathbf{C}^*$.

Small scale properties of $\mathcal{F}(U)$. As already noted, $\mathcal{F}(U)$ is a complex linear space for any open set $U \subset \mathbf{C}^*$. We shall make the convention that $\mathcal{F}(\emptyset) = \{0\}$, the trivial linear space. If U is not connected, we can write $U = U_1 \cap U_2$ where U_1, U_2 are two non empty disjoint open subsets, and it is obvious that a solution on U is determined by independant choices of a solution on U_1 and of a solution on U_2, *i.e.* there is an isomorphism:

$$\begin{cases} \mathcal{F}(U) \to \mathcal{F}(U_1) \times \mathcal{F}(U_2), \\ f \mapsto (f_{|U_1}, f_{|U_2}). \end{cases}$$

Whatever the number of connected components of U (even infinite), they are all open and a decomposition $U = \bigsqcup U_k$ gives rise to in a similar way an isomorphism $\mathcal{F}(U) \to \prod \mathcal{F}(U_k)$.

Therefore, the case of interest is if U is a non empty domain. In that case, we know for sure that the constants are solutions: $\mathbf{C} \subset \mathcal{F}(U)$, so that $\dim_{\mathbf{C}} \mathcal{F}(U) \geq 1$. Now, for a domain, one also has the upper bound: $\dim_{\mathbf{C}} \mathcal{F}(U) \leq 2$. This is a particular case of the "wronskian lemma", which will be proved in section 7.2, so we prefer not to give a proof here (but the reader can try to imagine one).

So the question is: for what kind of domain does one achieve the optimal dimension 2 ? We know that puncured neighborhood of 0 are excluded. So we choose $z_0 \neq 0$ and try for a power series $f(z) := \sum\limits_{n \geq 0} a_n (z - z_0)^n$. We find:

$$zf'' + f' = \sum_{n \geq 0} \left((n+1)^2 a_{n+1} + (n+1)(n+2) a_{n+2} z_0 \right) (z - z_0)^n$$

and deduce:

$$zf'' + f' = 0 \Longleftrightarrow \forall n \in \mathbf{N}\ ,\ (n+1)^2 a_{n+1} + (n+1)(n+2) a_{n+2} z_0 = 0.$$

The recurrence is readily solved: there is no condition on a_0, and $a_n = (a_1 z_0)\dfrac{(-1)^{n-1}}{n z_0^n}$ for $n \geq 1$. This means that $f = a_0 + (a_1 z_0)L$, where the series L has r.o.c. $|z_0|$. Actually, we recognize the series for the determination of the logarithm around z_0. It follows that, for any disk $U := \overset{\circ}{\mathrm{D}}(z_0, |z_0|)$, (see how it carefully avoids the singular point 0 ?) one has the optimal dimension $\dim_{\mathbf{C}} \mathcal{F}(U) = 2$. Also note that the image of $a_0 + (a_1 z_0)L$ by the initial condition map IC is $f \mapsto (a_0, a_1 z_0)$, so that it is an isomorphism in this case. (Finding out the equality of dimensions would not be by itself a sufficient argument.)

Large scale properties of $\mathcal{F}(U)$. Here, "large scale" means that we consider globally the collection of all open sets $U \subset \mathbf{C}^*$ and the collection of all linear spaces $\mathcal{F}(U)$. The main relation among these is that if $V \subset U$, there is a *restriction map* $f \mapsto f_{|V}$ from $\mathcal{F}(U)$ to $\mathcal{F}(V)$; of course, it is linear. The obvious fact that, if U is covered by open subsets V_k, then f can be uniquely recovered from the family of all the $f_k := f_{|V_k} \in \mathcal{F}(V_k)$, provided these are compatible (*i.e.* f_k and f_l have the same restriction in $\mathcal{F}(V_k \cap V_l)$) is formulated by saying that "$\mathcal{F}$ is a sheaf" (compete definition in section 7.4).

A property more related to the analyticity of the solutions is that, if U is a domain and if $V \subset U$ is a non empty open subset, then the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is injective: this is indeed a direct consequence of the principle of analytic continuation (theorem 3.1.6). As a consequence of this injectivity and of the calculation of dimensions, we find that if U is a disk centered at z_0 and if V is a domain, then the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is bijective. Therefore, for all domains U contained in some $\overset{\circ}{\mathrm{D}}(z_0, |z_0|)$, one has $\dim_{\mathbf{C}} \mathcal{F}(U) = 2$. Another consequence is that all the spaces $\mathcal{F}_{z_0}$ of germs of solution have dimension 2; and also that the initial contition map IC gives an isomorphism from $\mathcal{F}_{z_0}$ to $\mathbf{C}^2$.

Monodromy. We know the rules of the game. We fix $a, b \in \mathbf{C}^*$ and a path γ from a to b within $\mathbf{C}^*$. We cover the image curve of γ by disks $D_0, \ldots, D_n$ centered on the curve, the first one at a, the last one at b, and any two consecutive disks having non empty intersection. From this, we draw isomorphisms: first, $\mathcal{F}_a \leftarrow \mathcal{F}(D_0)$; then, for $k = 1, \ldots, n$: $\mathcal{F}(D_{k-1}) \to \mathcal{F}(D_{k-1} \cap D_k) \leftarrow \mathcal{F}(D_k)$; last, $\mathcal{F}(D_n) \to \mathcal{F}_b$. By composition, we get the isomorphism $\mathcal{F}_a \to \mathcal{F}_b$ induced by analytic continuation along the path γ. By the principle of monodromy (theorem 5.1.2), the isomorphism $\mathcal{F}_a \to \mathcal{F}_b$ depends only on the homotopy class of γ in $\mathbf{C}^*$. Taking $a = b$, this gives a map:

$$\pi_1(\mathbf{C}^*; a) \to \mathrm{GL}(\mathcal{F}_a).$$

From the algebraic rules stated in section 5.2, this is an anti-morphism of groups. However, in this particular case, we know that $\pi_1(\mathbf{C}^*; a) \simeq \mathbf{Z}$ is commutative, so an anti-morphism is the same thing as a morphism. Actually, this morphism is totally determined from the knowledge of the image of a generator of the fundamental group, for instance the homotopy class of the positive loop $\gamma : t \mapsto a e^{\mathrm{i}t}$ on $[0, 2\pi]$.

To see more concretely what this "monodromy representation" (it is its official name !) looks like, we shall take $a := 1$. Then a basis of $\mathcal{F}_1$ is given by (the germs of) the constant map 1 and the principal determination of the logarithm log. This gives an identification of $\mathrm{GL}(\mathcal{F}_1)$ with $\mathrm{GL}_2(\mathbf{C})$

and we obtain a second description of the monodromy representation, related to the first one by a commutative diagram:

$$\begin{tikzcd} \pi_1(\mathbf{C}^*;1) \arrow[r] \arrow[d] & \mathrm{GL}(\mathcal{F}_1) \arrow[d] \\ \mathbf{Z} \arrow[r] & \mathrm{GL}_2(\mathbf{C}) \end{tikzcd}$$

The vertical maps are isomorphisms, respectively induced by the choice of a generator of $\pi_1(\mathbf{C}^*;1)$ (the homotopy class of the loop γ) and by the choice of a basis of $\mathcal{F}_1$ (the germs of 1 and log). The horizontal maps are the monodromy representation in its abstract and in its matricial form. The lower horizontal map is characterized by the image of 1. This corresponds to the action of the loop γ. Along this loop, 1 is continuated to itself and log to $\log + 2i\pi$. Therefore, our basis $(1, \log)$ is transformed to $(1, \log + 2i\pi)$. The matrix of this automorphism is $\begin{pmatrix} 1 & 2i\pi \\ 0 & 1 \end{pmatrix}$. Therefore, we get at last the concrete description of the monodromy representation:

$$\begin{cases} \mathbf{Z} \to \mathrm{GL}_2(\mathbf{C}), \\ k \mapsto \begin{pmatrix} 1 & 2i\pi \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & 2i\pi k \\ 0 & 1 \end{pmatrix}. \end{cases}$$

Note that the monodromy automorphisms are unipotent: this is characteristic of the logarithm.

6.3 Return on the first example

We shall describe again what has been found in the first example (section 6.1) in the light of the constructions of section 6.2. Calling $\mathcal{F}_1$ the space of germs of solutions of $zf' = \alpha f$, where $\alpha \in \mathbf{C}$ is arbitrary, we saw that it is a linear space of dimension 1 and that turning around 0 once in the positive sense induced an automorphism $\psi : f \mapsto \beta f$ of $\mathcal{F}_1$, where $\beta := e^{2\alpha i\pi}$. Note that an automorphism of $\mathcal{F}_1$ is always of this form, that is, we have a canonical identification $\mathrm{GL}_1(\mathcal{F}_1) \simeq \mathbf{C}^*$. Here, "canonical" means that the isomorphism does not depend on the choice of a basis of $\mathcal{F}_1$. Now if we turn k times around 0, any $f \in \mathcal{F}(H_0)$ is multiplied k times by β, that is, the corresponding automorphism is $\psi^k : f \mapsto \beta^k f$. (This also works if $k < 0$.) In the light of the study of the second example, "turning around 0" just means performing analytic continuation along the loop $\gamma : t \mapsto e^{it}$ on $[0, 2\pi]$ and thereby identify $\pi_1(\mathbf{C}^*, 1)$ with $\mathbf{Z}$. Since we know that it is only the homotopy class of the loop that matters, and also that composing pathes, we must compose the automorphisms (using the rules of section 5.2), we get an anti-morphism of groups from $\pi_1(\mathbf{C}^*, 1)$ to $\mathrm{GL}(\mathcal{F}_1)$. As already noted, $\pi_1(\mathbf{C}^*, 1)$ being commutative, this is also a group morphism.

To the equation $zf' = \alpha f$, $\alpha \in \mathbf{C}$, we have therefore attached the monodromy representation:

$$\begin{cases} \mathbf{Z} \to \mathbf{C}^*, \\ k \mapsto \beta^k, \end{cases}$$

where $\beta := e^{2i\pi\alpha}$.

Chapter 7

Linear complex analytic differential equations

This chapter lays down the basic theory of linear complex analytic differential equations. (The nonlinear theory is much more difficult and we shall say nothing about it.) Since one of the ultimate goals of the whole theory is to understand the so-called "special functions", and since the study of these functions[1] shows interesting features of their asymptotic behaviour at infinity, the theory is done on the "Riemann sphere" which is the complex plane augmented with a "point at infinity". Therefore, the first section is a complement to the course on analytic functions.

7.1 The Riemann sphere

We want to study analytic functions "at infinity" as if this was a place, so that we can use geometric reasoning as we did in the complex plane $\mathbf{C}$. There are various ways[2] to do this. A most efficient one is to consider that there is a "point at infinity", that we write ∞ (without + or − sign). The resulting set is the *Riemann sphere*:

$$\mathbf{S} := \mathbf{C} \cup \{\infty\}.$$

Other names and notations for the same being are: the Alexandrov one-point compactification $\hat{\mathbf{C}}$; the complex projective line $\mathbf{P}^1(\mathbf{C})$. (There are some reasons to say that adding one point to the complex plane yields a complex *line* !) We shall not use these names and notations.

The reason why $\mathbf{S}$ is called a sphere can be understood through a process coming from cartography, the so-called "stereographic projection" from the north pole $N(0,0,1)$ of the sphere $X^2+Y^2+Z^2 = 1$ in $\mathbf{R}^3$ to its equatorial plane $P(Z=0)$. To any point $A(X,Y,Z)$ of $\dot{S} := S \setminus \{N\}$, it associates the intersection $B(x,y)$ of the straight line (NA) with P. Now we shall identify as usual B with $z := x + \mathrm{i}y \in \mathbf{C}$, whence a map $(X,Y,Z) \mapsto z$ from $\dot{S}$ to $\mathbf{C}$ given by the equation:

$$z = \frac{X}{1-Z} + \mathrm{i}\frac{Y}{1-Z}\cdot$$

[1]In this course, we shall not go in any detail about special functions. The best way to get acquainted to them is from far the book by Whitaker and Watson "A course of modern analysis", second part.

[2]For instance, in real projective geometry, one adds a whole (projective) line at infinity.

This map can easily be inverted into the map from $\mathbf{C}$ to $\dot{S}$ described by the equation:

$$z \mapsto \left(\frac{2\mathrm{Re}(z)}{|z|^2+1}, \frac{2\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right).$$

Exercice 7.1.1 Draw the corresponding figure and prove these equations.

So we have a homeomorphism (a bicontinuous bijection) of the plane $\mathbf{C}$ with $\dot{S}$; it is actually a diffeomorphism (differentiable with a differentiable inverse). Moreover, when z tends to infinity in the plane in the sense that $|z| \to +\infty$ (thus without any particular direction) then the corresponding point $A \in \dot{S}$ tends to N. For this reason, we transport the topology and differentiable structure on $S = \dot{S} \cup \{N\}$ to $\mathbf{S} = \mathbf{C} \cup \{\infty\}$. Therefore, $\mathbf{S}$ is homeomorphic to S. In particular, *the Riemann sphere is compact, arcwise connected and simply connected.*

The open subsets of $\mathbf{S}$ for this topology are, one the one hand the usual open subsets of $\mathbf{C}$; on the other hand, the open neighborhoods of ∞. These can be described as follows: they must contain a "disk centered at infinity" $\overset{\circ}{\mathrm{D}}(0,r) := \{z \in \mathbf{S} \mid |w| < r\}$, where $w := 1/z$ with of course the special convention that $1/\infty = 0$. There are three particular important open subsets of $\mathbf{S}$:

$$\mathbf{C}_0 := \mathbf{S} \setminus \{\infty\} = \mathbf{C}, \quad \mathbf{C}_\infty := \mathbf{S} \setminus \{0\}, \quad \mathbf{C}^* = \mathbf{S} \setminus \{0,\infty\} = \mathbf{C}_0 \cap \mathbf{C}_\infty.$$

When working with a point $z \in \mathbf{C}_0$, we may use the usual "coordinate" z; when working with a point $z \in \mathbf{C}_\infty$, we may use the new "coordinate" $w = 1/z$; when working with a point of $z \in \mathbf{C}^*$, we may use either coordinate. This works without problem for questions of topology and of differential calculus, because on the common domain of validity $\mathbf{C}^*$ of the two coordinates, the *changes of coordinates* $z \mapsto w = 1/z$ and $w \mapsto z = 1/w$ are homeomorphisms and even diffeomorphisms. But note that they are also both analytic, which justifies the following definition:

Definition 7.1.2 A function defined on a neighborhood of a point of $\mathbf{S}$ and with values in $\mathbf{C}$ is said to be *analytic*, resp. *holomorphic*, resp. *meromorphic* if it is so when expressed as a function of whichever coordinate (z or w) is defined at this point.

In the case of usual functions on $\mathbf{C}$, these definitions are obviously compatible with those given in chapter 3. Note that here again analyticity is equivalent to holomorphy. Practically, if one wants to study $f(z)$ at infinity, one puts $g(w) := f(1/w)$ and one studies g at $w = 0$.

Example 7.1.3 Let $f(z) := P(z)/Q(z) \in \mathbf{C}(z)$, $P, Q \neq 0$, be a non trivial rational function. We write $P(z) = a_0 + \cdots + a_d z^d$, with $a_d \neq 0$ (so that $\deg P = d$) and $Q(z) = b_0 + \cdots + b_e z^e$, with $b_e \neq 0$ (so that $\deg Q = e$). Moreover, we assume that P and Q have no common root. Then, f is holomorphic on $\mathbf{C} \setminus Q^{-1}(0)$ and meromorphic at points of $Q^{-1}(0)$: any root of multiplicity k of Q is a pole of order k of f. To see what happens at infinity, we use $w = 1/z$ and consider:

$$g(w) := f(1/w) = \frac{a_0 + \cdots + a_d w^{-d}}{b_0 + \cdots + b_e w^{-e}} = w^{e-d} \frac{a_0 w^d + \cdots + a_d}{b_0 w^e + \cdots + b_e}.$$

Since $a_d b_e \neq 0$, we conclude that, if $e \geq d$, then f is holomorphic at infinity; if $d > e$, then ∞ is a pole of order $d - e$. In any case, f is meromorphic on the whole of $\mathbf{S}$.
For f to be holomorphic on the whole of $\mathbf{S}$, it is necessary and sufficient that Q has no root, that is $e = 0$ (non constant complex polynomials always have roots); and that $e \geq d$, so $d = 0$. Therefore, a rational function is holomorphic on the whole of $\mathbf{S}$ if, and only if it is a constant.

Exercice 7.1.4 Study the rational $f_k(z) := \dfrac{z^k}{z^2-1}$ on $\mathbf{S}$ for $k \in \mathbf{N}$: what are its zeroes, its poles and their orders ? Whenever possible, do it using both coordinates z and w.

Example 7.1.5 To study the differential equation $zf' = \alpha f$ at infinity, we put $g(w) = f(w^{-1})$ so that $g'(w) = -f'(w^{-1})w^{-2} = -\alpha g(w)w^{-1}$, *i.e.* $wg' = -\alpha g$. The basic solution is, logically, $w^{-\alpha}$.

Exercice 7.1.6 What becomes the equation $zf'' + f' = 0$ at infinity ?

From the first example above, it follows that $\mathbf{C} \subset \mathcal{O}(\mathbf{S})$ and $\mathbf{C}(z) \subset \mathcal{M}(\mathbf{S})$. The following important theorem is proved in the books of Ahlfors, Cartan and Rudin:

Theorem 7.1.7 *(i) Every analytic function on* $\mathbf{S}$ *is constant:* $\mathcal{O}(\mathbf{S}) = \mathbf{C}$.
(ii) Every meromorphic function on $\mathbf{S}$ *is rational:* $\mathcal{M}(\mathbf{S}) = \mathbf{C}(z)$.

□

Description of holomorphic functions on some open sets. We now describe $\mathcal{O}(\Omega)$ for various open subsets Ω of $\mathbf{S}$.

1. $\mathcal{O}(\mathbf{C}_0)$: these are the power series $\sum_{n\geq 0} a_n z^n$ with an infinite r.o.c., for example e^z. They are called *entire functions*. A theorem of Liouville says that for an entire function f, if $|f| = O\left(|z|^k\right)$, then f is a polynomial. (This theorem and the second part of theorem 7.1.7 are easy consequences of the first part of theorem 7.1.7, which itself follows easily from the fact that every bounded entire function is constant. That last fact is less easy to prove; see the books by Ahlfors, Cartan, Rudin.)
2. $\mathcal{O}(\mathbf{C}_\infty)$: these are the functions $\sum_{n\geq 0} a_n z^{-n}$ where $\sum_{n\geq 0} a_n z^n$ is entire, for example $e^{1/z}$.
3. $\mathcal{O}(\mathbf{C}^*)$: these are "generalized Laurent series" of the form $\sum_{n\in\mathbf{Z}} a_n z^n$, where both $\sum_{n\geq 0} a_n z^n$ and $\sum_{n\geq 0} a_{-n} z^n$ are entire functions.
4. $\mathcal{O}(\overset{\circ}{\mathrm{D}}(0,R) \setminus \{0\})$: these are generalized Laurent series of the form $\sum_{n\in\mathbf{Z}} a_n z^n$, where $\sum_{n\geq 0} a_{-n} z^n$ is an entire function and $\sum_{n\geq 0} a_n z^n$ has r.o.c. R.
5. $\mathcal{O}(\overset{\circ}{\mathrm{D}}(\infty,r) \setminus \{\infty\})$: it is left to the reader to decribe this case from the previous one or from the case of an annulus tackled herebelow.

The two last items are particular cases ($r = 0$ or $R = +\infty$) of the *open annulus of radii* $r < R$:

$$\mathcal{C}(r,R) := \overset{\circ}{\mathrm{D}}(0,R) \setminus \overline{\mathrm{D}}(0,r) = \{z \in \mathbf{C} \mid r < |z| < R\}.$$

Then the elements of $\mathcal{O}\left(\mathcal{C}(r,R)\right)$ are generalized Laurent series of the form $\sum_{n\in\mathbf{Z}} a_n z^n$, where $\sum_{n\geq 0} a_{-n} z^n$ has r.o.c. $1/r$ and $\sum_{n\geq 0} a_n z^n$ has r.o.c. R.

Note that the apparition of generalized Laurent series here means that essential singularities are possible. For functions having poles at 0 or ∞, the corresponding series will be usual Laurent series. All the facts above are proved in the books by Ahlfors, Cartan and Rudin.

7.2 Equations of order n and systems of rank n

Let Ω be a non empty connected subset (a domain) of $\mathbf{S}$ and let $a_1, \dots, a_n \in \mathcal{O}(\Omega)$. We shall call denote $E_{\underline{a}}$ the following *linear homogeneous scalar differential equation of order n*:

$$f^{(n)} + a_1 f^{(n-1)} + \cdots + a_n f = 0.$$

The reader may think of the two examples $f' - (\alpha/z)f = 0$ and $f'' + (1/z)f' + 0f = 0$ of chapter 6; there, $\Omega = \mathbf{C}^*$. For any open subset U of Ω, we write $\mathcal{F}_{\underline{a}}(U)$ the set of solutions of $E_{\underline{a}}$ on U:

$$\mathcal{F}_{\underline{a}}(U) := \{f \in \mathcal{O}(U) \mid \forall z \in U\ ,\ f^{(n)}(z) + a_1(z) f^{(n-1)}(z) + \cdots + a_n(z) f(z) = 0\}.$$

It is a linear space over $\mathbf{C}$. The reason to consider various open sets U is that we hope to understand better the differential equation $E_{\underline{a}}$ on the whole of Ω by studying the collection of all spaces $\mathcal{F}_{\underline{a}}(U)$. For instance, we saw in chapter 6 examples where $\mathcal{F}_{\underline{a}}(\Omega) = \{0\}$ but nevertheless some local solutions are interesting.

By convention, we agree that $\mathcal{F}_{\underline{a}}(\emptyset) = \{0\}$, the trivial space. If U is not connected and if $U = \bigsqcup U_i$ is its decomposition in connected components, we know that the U_i are open sets and that there is an isomorphism $\mathcal{O}(U) \to \prod \mathcal{O}(U_i)$, which sends $f \in \mathcal{O}(U)$ to the family (f_i) of its restrictions $f_i := f_{|U_i}$. This clearly induces an isomorphism of $\mathcal{F}_{\underline{a}}(U)$ with $\prod \mathcal{F}_{\underline{a}}(U_i)$. For these reasons, we shall most of the time assume that U is a non empty domain.

Let $A = (a_{i,j})_{1 \leq i,j \leq n} \in \mathrm{Mat}_n(\Omega)$ be a matrix having analytic coefficients $a_{i,j} \in \mathcal{O}(\Omega)$. We shall call denote S_A the following *linear homogeneous vectorial differential equation of rank n and order* 1:

$$X' = AX.$$

The reader may think of the vectorial form of the equation of the logarithm in chapter 6; there, $\Omega = \mathbf{C}^*$.

Remark 7.2.1 There may be a difficulty when $\infty \in \Omega$. Indeed, then the coordinate $w := 1/z$ should be used. Writing $B(w) := A(1/w)$ and $Y(w) := X(1/w)$, S_A becomes: $Y'(w) = -w^{-2}B(w)Y(W)$. However, it is not true that if A is analytic at ∞ then $-w^{-2}B(w)$ is analytic at $w = 0$. For instance, the differential equation $f' = f$ is not analytic at ∞, despite the apparences. The totally rigorous solution to this difficulty is to write S_A in the form $dX = Adz$ and to require that the "matrix of differential forms" $A(z)dz$ be analytic on Ω. At ∞ that would translate to the correct condition that $-w^{-2}B(w)$ is analytic at $w = 0$. Except in the following exercice, we shall not tackle this problem anymore, except in concrete cases when we have to.

Exercice 7.2.2 Prove that a differential system that is holomorphic on the whole of $\mathbf{S}$ has the form $X' = 0$.

For any open subset U of Ω, we write $\mathcal{F}_A(U)$ the set of vectorial solutions of S_A on U:

$$\mathcal{F}_{\underline{a}}(U) := \{X \in O(U)^n \mid \forall z \in U \ , \ X'(z) = A(z)X(z)\}.$$

It is a linear space over $\mathbf{C}$. By convention, we agree that $\mathcal{F}_A(\emptyset) = \{0\}$. If $U = \bigsqcup U_i$ is the decomposition of U in connected components, the map which sends $X \in \mathcal{F}_A(U)$ to the family of its restrictions $X_{|U_i}$ is an isomorphism of $\mathcal{F}_{\underline{a}}(U)$ with $\prod \mathcal{F}_{\underline{a}}(U_i)$. We shall also sometimes consider *matricial* solutions $M \in \mathrm{Mat}_{n,p}(O(U))$, that is matrices with coefficients in $O(U)$ and such that $M'(z) = A(z)M(z)$ on U; it is equivalent to say that the columns of M belong to $\mathcal{F}_A(U)$. For instance, if $A \in \mathrm{Mat}_n(\mathbf{C})$ (constant coefficients), then e^{zA} is a matricial solution of S_A. Note that S_A can be seen as a system of scalar differential equations; if $f_1, \ldots, f_n$ are the components of X, then:

$$S_A \Longleftrightarrow \begin{cases} f_1' = a_{1,1}f_1 + \cdots + a_{1,n}f_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ f_n' = a_{n,1}f_1 + \cdots + a_{n,n}f_n. \end{cases}$$

Proposition 7.2.3 *Given $a_1, \ldots, a_n \in O(\Omega)$ and $f \in O(U)$, $U \subset \Omega$, define:*

$$A_{\underline{a}} := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_2 & -a_1 \end{pmatrix} \in Mat(O(\Omega)) \quad \text{and} \quad X_f := \begin{pmatrix} f \\ f' \\ \vdots \\ f^{(n-1)} \end{pmatrix} \in O(U)^n.$$

Then, the scalar differential equation $E_{\underline{a}}$ is equivalent to the vectorial differential equation S_A for $A := A_{\underline{a}}$ in the following sense: for any open subset $U \subset \Omega$, the map $f \mapsto X_f$ is an isomorphism of linear spaces from $\mathcal{F}_{\underline{a}}(U)$ to $\mathcal{F}_A(U)$.

Proof. - Let $X \in O(U)^n$ have components $f_1, \ldots, f_n$. Then:

$$X' = A_{\underline{a}}X \Longleftrightarrow f_1' = f_2, \ldots, f_{n-1}' = f_n \text{ and } f_n' + a_1 f_n + \cdots + a_n f_1 = 0 \Longleftrightarrow X = X_f,$$

where $f := f_1$ is a solution of $E_{\underline{a}}$ and the unique antecedent of X. □

The Wronskian.

Definition 7.2.4 The *wronskian matrix* of $f_1, \ldots, f_n \in O(U)$ is:

$$W_n(f_1, \ldots, f_n) := \begin{pmatrix} f_1 & \ldots & f_j & \ldots & f_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ f_1^{(i-1)} & \ldots & f_j^{(i-1)} & \ldots & f_n^{(i-1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & \ldots & f_j^{(n-1)} & \ldots & f_n^{(n-1)} \end{pmatrix} = [X_{f_1}, \ldots, X_{f_n}] \in \mathrm{Mat}_n(O(U)).$$

Beware that its (i, j)-coefficient is $f_j^{(i-1)}$. The *wronskian determinant* of $f_1, \ldots, f_n \in O(U)$ is:

$$w_n(f_1, \ldots, f_n) := \det W_n(f_1, \ldots, f_n) \in O(U).$$

It is also simply called the *wronskian* of $f_1, \ldots, f_n$.

The wronskian is an extremely useful tool in the study of differential systems. We prepare its computation with two lemmas.

Lemma 7.2.5 *Let $X_1, \ldots, X_n : U \to \mathbf{C}^n$ be vector valued analytic functions. Then $\det(X_1, \ldots, X_n) \in \mathcal{O}(U)$ and:*

$$(\det(X_1, \ldots, X_n))' = \det(X_1', \ldots, X_i, \ldots, X_n) + \cdots + \det(X_1, \ldots, X_i', \ldots, X_n) + \cdots + \det(X_1, \ldots, X_i, \ldots, X_n').$$

Proof. - From the "small" Leibniz formula $(fg)' = f'g + fg'$, one draws by induction the "big" Leibniz formula $(f_1 \cdots f_n)' = f_1' \cdots f_i \cdots f_n + \cdots + f_1 \cdots f_i' \cdots f_n + \cdots + f_1 \cdots f_i \cdots f_n'$. Then we apply this formula to each of the $n!$ monomials composing the determinant. □

Lemma 7.2.6 *Let $X_1, \ldots, X_n \in \mathbf{C}^n$ and let $A \in \mathrm{Mat}_n(\mathbf{C})$. Then:*

$$\det(AX_1, \ldots, X_i, \ldots, X_n) + \cdots + \det(X_1, \ldots, AX_i, \ldots, X_n) + \cdots + \det(X_1, \ldots, X_i, \ldots, AX_n) = \mathrm{Tr}(A) \det(X_1, \ldots, X_n).$$

Proof. - The left hand side is an alternated n-linear function of the X_i, so, by general multilinear algebra (see for instance Lang's "Algebra"), it is equal to $C \det(X_1, \ldots, X_n)$ for some constant C. Taking for $X_1, \ldots, X_n$ the canonical basis of $\mathbf{C}^n$ gives the desired result. □

Proposition 7.2.7 *Let $X_1, \ldots, X_n \in \mathcal{F}_A(U)$. Write $\mathcal{X} := [X_1, \ldots, X_n]$ the matrix having the X_i as columns. Then $\mathcal{X} \in \mathrm{Mat}_n(\mathcal{O}(U))$, $\mathcal{X}' = A\mathcal{X}$ and:*

$$(\det \mathcal{X})' = \mathrm{Tr}(A)(\det \mathcal{X}).$$

As a consequence, if U is a domain, either $\det \mathcal{X}$ vanishes nowhere on U, or it vanishes identically.

Proof. - The fact that $\mathcal{X} \in \mathrm{Mat}_n(\mathcal{O}(U))$ is obvious; the fact that $\mathcal{X}' = A\mathcal{X}$ follows because the columns of $A\mathcal{X}$ are the $AX_i = X_i'$. For the last formula, we compute with the help of the two lemmas above:

$$\begin{aligned}
(\det \mathcal{X})' &= \det(X_1', \ldots, X_i, \ldots, X_n) + \cdots + \det(X_1, \ldots, X_i', \ldots, X_n) + \cdots + \det(X_1, \ldots, X_i, \ldots, X_n') \\
&= \det(AX_1, \ldots, X_i, \ldots, X_n) + \cdots + \det(X_1, \ldots, AX_i, \ldots, X_n) + \cdots + \det(X_1, \ldots, X_i, \ldots, AX_n) \\
&= \mathrm{Tr}(A) \det(X_1, \ldots, X_n) = \mathrm{Tr}(A)(\det \mathcal{X}).
\end{aligned}$$

The last statement is proved as follows. Write for short $w(z) := (\det \mathcal{X})(z)$ and suppose that $w(z_0) = 0$ for some $z_0 \in U$. Let V be a non empty simply connected open neighborhood of z_0 in U (for instance, an open disk centered at z_0). Then the analytic function $\mathrm{Tr}(A)$ has a primitive f on V (see section 3.3). Therefore, $w' = f'w$ on V, that is $(e^{-f}w)' = 0$ so that $w = Ce^f$ for some constant $C \in \mathbf{C}$. Since $w(z_0) = 0$, this implies that $C = 0$ and w vanishes on V. By the principle of analytic continuation (theorem 3.1.6), U being connected, w vanishes on U. □

Corollary 7.2.8 *Let $f_1, \ldots, f_n \in \mathcal{F}_{\underline{a}}(U)$ and write $w(z) := w_n(f_1, \ldots, f_n)(z)$. Then $w \in \mathcal{O}(U)$, $w' = -a_1 w$ and, if U is a domain, either w vanishes nowhere on U, or it vanishes identically.*

Proof. - Indeed, $\mathrm{Tr}(A_{\underline{a}}) = -a_1$. □

Definition 7.2.9 Let U be a domain.
(i) If $X_1, \ldots, X_n \in \mathcal{F}_A(U)$ are such that $\det(X_1, \ldots, X_n)$ vanishes nowhere on U, then $\mathcal{X} := [X_1, \ldots, X_n]$ is called a *fundamental matricial solution of S_A on U*.
(ii) If $f_1, \ldots, f_n \in \mathcal{F}_{\underline{a}}(U)$ are such that their wronskian vanishes nowhere on U, then $(f_1, \ldots, f_n)$ is called a *fundamental system of solutions of $E_{\underline{a}}$ on U*.

Examples 7.2.10 (i) If $(f_1, \ldots, f_n)$ is a fundamental system of solutions of $E_{\underline{a}}$ on U, then $[\mathcal{X}_{f_1}, \ldots, \mathcal{X}_{f_n}]$ is a fundamental matricial solution of S_A on U for $A := A_{\underline{a}}$.
(ii) If $A \in \mathrm{Mat}_n(\mathbf{C})$, then e^{zA} is a fundamental matricial solution of $X' = AX$ on $\mathbf{C}$.
(iii) If $\alpha \in \mathbf{C}$, then z^α is a fundamental matricial solution (of rank 1) and also a fundamental system of solutions of $zf' = \alpha f$ on $\mathbf{C} \setminus \mathbf{R}_-$.
(iv) The pair $(1, \log)$ is a fundamental system of solutions of $zf'' + f' = 0$ on $\mathbf{C} \setminus \mathbf{R}_-$. Indeed, its wronskian matrix is $\begin{pmatrix} 1 & \log z \\ 0 & 1/z \end{pmatrix}$. Its wronskian determinant is $w(z) = 1/z$, which satisfies $w' = -a_1 w$ with here $a_1 = 1/z$ (and $a_2 = 0$).

Remark 7.2.11 If $\mathcal{X} \in \mathrm{Mat}_n(\mathcal{O}(U))$ is a fundamental matricial solution of S_A, then $\mathcal{X}^{-1} \in \mathrm{Mat}_n(\mathcal{O}(U))$. Indeed, the inverse of an arbitrary matrix A is computed as $\dfrac{1}{\det A}\, {}^t\mathrm{com}(A)$, where $\mathrm{com}(A)$ (the so-called "comatrix" of A) has as coefficients the minor determinants of A. In our case, $\mathrm{com}(\mathcal{X})$ and its transpose ${}^t\mathrm{com}(\mathcal{X})$ are obviously in $\mathrm{Mat}_n(\mathcal{O}(U))$; and, since $\det \mathcal{X}$ is analytic and vanishes nowhere, $\dfrac{1}{\det X} \in \mathcal{O}(U)$. We shall therefore write $\mathcal{X} \in \mathrm{GL}_n(\mathcal{O}(U))$. More generally, for any commutative ring R, the matrices $A \in \mathrm{Mat}_n(R)$ which have an inverse $A^{-1} \in \mathrm{Mat}_n(R)$ are those such that $\det A$ is invertible in R. The group of such matrices is written $\mathrm{GL}_n(R)$.

Theorem 7.2.12 *Let U be a non empty domain. Let $X_1, \ldots, X_n \in \mathcal{F}_A(U)$ and let $\mathcal{X} := [X_1, \ldots, X_n] \in \mathrm{Mat}_n(\mathcal{O}(U))$. Write $w(z) := (\det \mathcal{X})(z)$. If there exists $z_0 \in U$ such that $w(z_0) \neq 0$, then w vanishes nowhere (and therefore $\mathcal{X}$ is a fundamental matricial solution of S_A on U). In that case, $(X_1, \ldots, X_n)$ is a basis of $\mathcal{F}_A(U)$.*

Proof. - Suppose that there exists $z_0 \in U$ such that $w(z_0) \neq 0$. The fact that w vanishes nowhere is contained in the previous proposition. Then $\mathcal{X}$ is invertible. Let $X \in \mathcal{F}_A(U)$ be an arbitrary solution. Taking $Y := \mathcal{X}^{-1}X$, we can write $X = \mathcal{X}Y$ with $Y \in \mathcal{O}(U)^n$. From $X' = AX$ we draw:

$$(\mathcal{X}Y)' = A(\mathcal{X}Y) \Longrightarrow \mathcal{X}Y' + \mathcal{X}'Y = A\mathcal{X}Y \Longrightarrow \mathcal{X}Y' + A\mathcal{X}Y = A\mathcal{X}Y \Longrightarrow \mathcal{X}Y' = 0 \Longrightarrow Y' = 0,$$

so that $Y \in \mathbf{C}^n$ since U is connected (otherwise, having a zero derivative would not imply being constant). If $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$ are the components of Y, we conclude that $X = \lambda_1 X_1 + \cdots + \lambda_n X_n$, and $(X_1, \ldots, X_n)$ is a generating system. The same computation shows that the coefficients $\lambda_1, \ldots, \lambda_n$ are unique, so that $(X_1, \ldots, X_n)$ is indeed a basis of $\mathcal{F}_A(U)$. $\Box$

Remark 7.2.13 If $w = 0$, then it is clear that, for all z_0 in U, the vectors $X_1(z_0), \ldots, X_n(z_0)$ are linearly dependent over $\mathbf{C}$. One can prove algebraically the much stronger statement that $X_1, \ldots, X_n$ are linearly dependent over $\mathbf{C}$ (with coefficients that do not depend on z): this is the "wronskian lemma" (lemma 1.12 in the book by van der Put and Singer "Galois theory of linear differential equations").

Corollary 7.2.14 *If S_A admits a fundamental matricial solution $\mathcal{X} = [X_1, \ldots, X_n]$, then, for any basis $(Y_1, \ldots, Y_n)$ of $\mathcal{F}_A(U)$, the matrix $\mathcal{Y} := [Y_1, \ldots, Y_n]$ is a fundamental matricial solution.*

Proof. - We write $Y_i = \mathcal{X}P_i$, with $P_i \in \mathbf{C}^n$ as in the theorem. Then, putting $P := [P_1, \ldots, P_n]$, one has $\mathcal{Y} = \mathcal{X}P$ with $P \in \mathrm{Mat}_n(\mathbf{C})$. Since P links two bases, $\det P \neq 0$. Thus $\det \mathcal{Y} = (\det \mathcal{X})(\det P)$ does not vanish. □

Corollary 7.2.15 *Let U be a non empty domain. Let $f_1, \ldots, f_n \in \mathcal{F}_{\underline{a}}(U)$. Write $w(z) := (w_n(f_1, \ldots, f_n))(z)$. If there exists $z_0 \in U$ such that $w(z_0) \neq 0$, then w vanishes nowhere and $(f_1, \ldots, f_n)$ is a basis of $\mathcal{F}_A(U)$. In that case, any basis of $\mathcal{F}_{\underline{a}}(U)$ is a fundamental system of solutions of $E_{\underline{a}}$.*

Proof. - This just uses the isomorphism $f \mapsto X_f$ from $\mathcal{F}_{\underline{a}}(U)$ to $\mathcal{F}_A(U)$, where $A := A_{\underline{a}}$ (proposition 7.2.3). □

As the following result shows, the existence of a fundamental matricial solution or a fundamental system of solutions actually corresponds to an "optimal case"[3], where the solution space has dimension n.

Theorem 7.2.16 *For any domain $U \subset \Omega$, one has $\dim_{\mathbf{C}} \mathcal{F}_A(U) \leq n$.*

Proof. - This can be deduced from the "wronskian lemma" (see remark 7.2.13), but we shall do it using linear algebra over the field $K := \mathcal{M}(U)$. Since elements of $\mathcal{F}_A(U)$ belong to K^n, the maximum number of K-linearly independent elements $X_1, \ldots, X_k \in \mathcal{F}_A(U)$ is some integer $k \leq n$. If $k = 0$, then $\mathcal{F}_A(U) = \{0\}$ and the result is trivial, so assume $k \geq 1$. Choose such elements $X_1, \ldots, X_k$; we are going to prove that they form a basis of the $\mathbf{C}$-linear space $\mathcal{F}_A(U)$ and the conclusion will follow.
Since $X_1, \ldots, X_k$ are K-linearly independent elements, they are plainly $\mathbf{C}$-linearly independent elements. Now let $X \in \mathcal{F}_A(U)$. By the maximality property of $X_1, \ldots, X_k$, one can write $X = f_1X_1 + \cdots + f_kX_k$, with $f_1, \ldots f_k \in K = \mathcal{M}(U)$. Derivating this relation and using S_A, we find:

$$\begin{aligned}
X' = f_1'X_1 + \cdots + f_k'X_k + f_1X_1' + \cdots + f_kX_k' &\Longrightarrow AX = f_1'X_1 + \cdots + f_k'X_k + f_1AX_1 + \cdots + f_kAX_k \\
&\Longrightarrow AX = f_1'X_1 + \cdots + f_k'X_k + A(f_1X_1 + \cdots + f_kX_k) \\
&\Longrightarrow f_1'X_1 + \cdots + f_k'X_k = 0 \\
&\Longrightarrow f_1' = \cdots = f_k' = 0,
\end{aligned}$$

since $X_1, \ldots, X_k$ are K-linearly independent; this implies $f_1, \ldots f_k \in \mathbf{C}$. □

Corollary 7.2.17 *For any domain $U \subset \Omega$, one has $\dim_{\mathbf{C}} \mathcal{F}_{\underline{a}}(U) \leq n$.*

Proof. - Again use the isomorphism $f \mapsto X_f$ from $\mathcal{F}_{\underline{a}}(U)$ to $\mathcal{F}_A(U)$, where $A := A_{\underline{a}}$ (proposition 7.2.3). □

Exercice 7.2.18 Prove the wronskian lemma mentioned in remark 7.2.13 and deduce from it theorem 7.2.16.

[3] In old litterature on functional equations, the authors said that they had a "full complement of solutions" when they reached the maximum reasonable number of independant solutions.

7.3 The existence theorem of Cauchy

From now on, we shall state and prove theorems for systems S_A and leave it to the reader to translate them into results about equations $E_{\underline{a}}$. Also, we shall call S_A a differential equation, without saying explicitly "vectorial".

Lemma 7.3.1 *(i) Let $A \in Mat_n(\mathbf{C}[[z]])$ be a matrix with coefficients in $\mathbf{C}[[z]]$. Then there is a unique $\mathcal{X} \in Mat_n(\mathbf{C}[[z]])$ such that $\mathcal{X}' = A\mathcal{X}$ and $\mathcal{X}(0) = I_n$. Moreover, $\mathcal{X} \in GL_n(\mathbf{C}[[z]])$.*
(ii) Suppose that A has a strictly positive radius of convergence, i.e. *$A \in Mat_n(\mathbf{C}\{z\})$. Then $\mathcal{X}$ also has a strictly positive radius of convergence,* i.e. *$X \in GL_n(\mathbf{C}\{z\})$.*

Proof. - (i) Write $A = A_0 + zA_1 + \cdots$ and $\mathcal{X} = \mathcal{X}_0 + z\mathcal{X}_1 + \cdots$. Then:

$$\begin{cases} \mathcal{X}' = A\mathcal{X}, \\ \mathcal{X}(0) = I_n \end{cases} \iff \begin{cases} \mathcal{X}_0 = I_n, \\ \forall k \geq 0\ ,\ (k+1)\mathcal{X}_{k+1} = A_0\mathcal{X}_k + \cdots + A_k\mathcal{X}_0, \end{cases}$$

so that it can be recursively solved and admits a unique solution. Then $(\det \mathcal{X})(0) = 1$, so that $\det \mathcal{X}$ is invertible in $\mathbf{C}[[z]]$; as already noted, this implies $\mathcal{X} \in \mathrm{GL}_n(\mathbf{C}[[z]])$.
(ii) We use the same norm as in section 1.4. Choose $R > 0$ and strictly smaller than the r.o.c. of A. Then the $|||A_k|||R^k$ tend to 0 as $k \to +\infty$, so that they are bounded: there exists $C > 0$ such that $|||A_k||| \leq CR^{-k}$ for all $k \geq 0$. If necessary, reduce R so that moreover $CR \leq 1$. We are going to prove that $|||\mathcal{X}_k||| \leq R^{-k}$ for all $k \geq 0$ which will give the conclusion. The inequality is obvious for $k = 0$. Assume it for all indexes up to k. Then:

$$||\mathcal{X}_{k+1}|| \leq \frac{1}{k+1} \sum_{i+j=k} ||A_i||\,||\mathcal{X}_j|| \leq \frac{1}{k+1} \sum_{i+j=k} (CR^{-i})(R^{-j}) = (CR)R^{-(k+1)} \leq R^{-(k+1)}.$$

□

Theorem 7.3.2 (Cauchy existence theorem) *Let $A \in Mat_n(\Omega)$. Then, for all $z_0 \in \Omega$, there exists a domain $U \subset \Omega$, $U \ni z_0$ such that the map $X \mapsto X(z_0)$ from $\mathcal{F}_A(U)$ to $\mathbf{C}^n$ is an isomorphism. In other words, on such a domain, the Cauchy problem $\begin{cases} X' = AX, \\ X(z_0) = X_0 \end{cases}$ admits a unique solution for any initial condition $X_0 \in \mathbf{C}^n$.*

Proof. - Suppose for a moment that we put $B(z) := A(z_0 + z)$ and $Y(z) := X(z_0 + z)$. Then the Cauchy problem $\begin{cases} X' = AX, \\ X(z_0) = X_0 \end{cases}$ is equivalent to $\begin{cases} Y' = AY, \\ Y(0) = X_0 \end{cases}$. In other words, we may (and shall) assume from the beginning that $z_0 = 0$.
From the previous lemma, there exists $\mathcal{X} \in \mathrm{GL}_n(\mathbf{C}\{z\})$ such that $\mathcal{X}' = A\mathcal{X}$ and $\mathcal{X}(0) = I_n$. Let $U \subset \Omega$ be any non empty domain containing 0 on which $\mathcal{X}$ converges and $\det \mathcal{X}$ does not vanish. Then, we may look for X in the form $\mathcal{X}Y$. With the same computation as in theorem 7.2.12, we find that our Cauchy problem is equivalent to $\begin{cases} Y' = 0, \\ Y(0) = X_0 \end{cases}$, that is to $Y = X_0$. Therefore, its unique solution on U is $\mathcal{X}X_0$. □

Corollary 7.3.3 *For all $z_0 \in \Omega$, there exists a domain $U \subset \Omega$, $U \ni z_0$, such that* $\dim_{\mathbf{C}} \mathcal{F}_A(U) = n$.

Corollary 7.3.4 *There are no singular solutions on any open subset of Ω.*

Proof. - Suppose X is a solution in a punctured disk $\overset{\circ}{\mathrm{D}}(z_0,R) \setminus \{z_0\}$, where $\overset{\circ}{\mathrm{D}}(z_0,R) \subset \Omega$ and $R > 0$. Let $\mathcal{X}$ be a fundamental solution on $\overset{\circ}{\mathrm{D}}(z_0,R') \subset \overset{\circ}{\mathrm{D}}(z_0,R)$, $R' > 0$. Then $X = \mathcal{X}X_0$ on $\overset{\circ}{\mathrm{D}}(z_0,R')$, with $X_0 \in \mathbf{C}^n$; thus z_0 is an inexistent singularity of X. $\square$

7.4 The sheaf of solutions

Let Ω be a non empty domain of $\mathbf{S}$, $a_1,\ldots,a_n \in O(\Omega)$ and $A \in \mathrm{Mat}_n(O(\Omega))$. Remember that we have denoted $\mathcal{F}_{\underline{a}}(U)$, resp. $\mathcal{F}_A(U)$, the complex linear space of solutions of $E_{\underline{a}}$, resp. S_A on an open subset $U \subset \Omega$. In this section, we shall study at the same time some topological and algebraic properties of the maps $\mathcal{F}_{\underline{a}} : U \mapsto \mathcal{F}_{\underline{a}}(U)$ and $\mathcal{F}_A : U \mapsto \mathcal{F}_A(U)$. To that end, we shall indifferently write $\mathcal{F}$ either for $\mathcal{F}_{\underline{a}}$ or for $\mathcal{F}_A$.

Sheaves. The first important property is that $\mathcal{F}$ is a *sheaf.* (For the general theory, see the book by Godement, "Topologie algébrique et théorie des faisceaux; the book of Ahlfors uses a less flexible presentation, resting on "espaces étalés".) To be precise, $\mathcal{F}$ is a *sheaf of complex linear spaces over* Ω, meaning that it associates to every open subset $U \subset \Omega$ a $\mathbf{C}$-linear space $\mathcal{F}(U)$, with the following extra structure and conditions (which are of course completely obvious in the case of $\mathcal{F}_{\underline{a}}$ and $\mathcal{F}_A$):

1. If $V \subset U$ are two open subsets of Ω, there is a morphism (linear map) of *restriction* $\mathcal{F}(U) \to \mathcal{F}(V)$. In general, an element of $\mathcal{F}(U)$ is called a "section" over U (in our cases of interest, sections are solutions f or X) and we shall write $s_{|V}$ its restriction to V. The restriction maps must satisfy natural compatibility conditions: if $V = U$, it is the identity map of $\mathcal{F}(U)$; if $W \subset V \subset U$, then the restriction map $\mathcal{F}(U) \to \mathcal{F}(W)$ is the composite of the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ and $\mathcal{F}(V) \to \mathcal{F}(W)$.

2. Given an open covering $U = \bigcup U_i$ of the open subset $U \subset \Omega$ by open subsets $U_i \subset U$, there arises a map $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$, defined as $s \mapsto (s_i)$ where $s \in \mathcal{F}(U)$ and the s_i are the restrictions $s_{|U_i}$. Then, the second requirement is that the map be injective: a section is totally determined by its restrictions to an open covering.

3. In the previous construction, it is a consequence of the compatibility condition stated before that one has for all i,j the equality $(s_i)_{|U_i \cap U_j} = (s_j)_{|U_i \cap U_j}$ on $U_i \cap U_j$. Our last requirement is that conversely, for every open covering $U = \bigcup U_i$ and for every family of sections $(s_i) \in \prod \mathcal{F}(U_i)$, if this family satisfies the compatibility condition that for all i,j one has the equality $(s_i)_{|U_i \cap U_j} = (s_j)_{|U_i \cap U_j}$, then there exists a section $s \in \mathcal{F}(U)$ such that, for all i, one has $s_i = s_{|U_i}$. The section s is of course unique, because of the second requirement.

Local systems. We will not give here the precise and general definition of a "local system" of linear spaces, but rather state the properties of $\mathcal{F} = \mathcal{F}_{\underline{a}}$ or $\mathcal{F}_A$ which imply that it is indeed a local system; and then, draw the consequences, using direct arguments (taking in account that our "sections" are actually functions or vectors of functions). But most of what we are going to do until the end of this chapter is valid in a much more general form, and the reader *should* look at the beginning of the extraordinary book of Deligne "Équations différentielles à points singuliers réguliers" to see the formalism. The basic fact is the following.

Proposition 7.4.1 *Every $a \in \Omega$ has a neighborhood $U \subset \Omega$ on which the sheaf $\mathcal{F}$ is "constant", meaning that, for every non empty domain $V \subset U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is bijective.*

Proof. - Choose for U the disk of convergence of a fundamental system of solutions (case $\mathcal{F} = \mathcal{F}_{\underline{a}}$) or a fundamental matricial solution (case $\mathcal{F} = \mathcal{F}_A$); if necessary, shrink the disk so that $U \subset \Omega$. Then we know that $\dim_{\mathbf{C}} \mathcal{F}(U) = n$ (section 7.3). Now, for every non empty domain $V \subset U$, we know that the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is injective (principle of analytic continuation) and that $\dim_{\mathbf{C}} \mathcal{F}(V) \leq n$ (theorem 7.2.16). $\square$

Germs. We defined germs of functions and germs of solutions, but germs can actually be defined for any sheaf $\mathcal{F}$ on Ω as follows. Call "local element" of $\mathcal{F}$ a pair (U,s) where U is an open subset of Ω and $s \in \mathcal{F}(U)$. Fix a point $a \in \Omega$. We say that two local elements (U_1,s_1) and (U_2,s_2) define the same germ at a if there exists an open neighborhood V of a such that $V \subset U_1 \cap U_2$ and $(s_1)_{|V} = (s_2)_{|V}$. The germ defined by a local element (U,s) at $a \in U$ will be denoted s_a and called *the germ of s at a*. The set of all germs at a is written $\mathcal{F}_a$. (Do not confuse this notation with that of $\mathcal{F}_{\underline{a}}$!) In our case (sheaf of linear spaces), germs can be added and multiplied by scalars, so that $\mathcal{F}_a$ is actually a linear space. Moreover, there is for $a \in U$ a natural map $\mathcal{F}(U) \to \mathcal{F}_a$ and it is of course a linear map. Now, our sheaves being local systems, we get the following consequence of the previous proposition:

Corollary 7.4.2 *For all sufficiently small connected neighborhoods of a, the linear maps $\mathcal{F}(U) \to \mathcal{F}_a$ are bijective.*

In the particular case of $\mathcal{F}_{\underline{a}}$ and $\mathcal{F}_A$, there is an additional structure and property that do not make sense for general local systems. Indeed, there is an "initial condition" map form $\mathcal{F}(U)$ to $\mathbf{C}^n$, which sends $f \in \mathcal{F}_{\underline{a}}(U)$ to $(f(a),\dots,f^{(n-1)}(a))$ and $X \in \mathcal{F}_A(U)$ to $X(a)$. We know from Cauchy theorem that it is bijective for all sufficiently small connected neighborhoods of a.

Corollary 7.4.3 *The initial condition map induces an isomorphism $\mathcal{F}_a \mapsto \mathbf{C}^n$.*

Exercice 7.4.4 Check that $U \mapsto O(U)$ is a sheaf of $\mathbf{C}$-algebras, and that the $\mathbf{C}$-algebra of germs O_a is $\mathbf{C}\{z-a\}$. (However, beware that O is not a local system.)

7.5 The monodromy representation

Monodromy. We shall play the usual game but described in a more general guise. (For an even greater generality, see the book of Deligne.) To begin with, we fix $a,b \in \Omega$ and a path γ from a to b in Ω. We cover the image curve by "sufficiently small" disks $D_0,\dots,D_N$ in such a way that $a \in D_0$, $b \in D_N$ and, for $i = 1,\dots,N$, $D_i \cap D_{i-1} \neq \emptyset$. Then there are isomorphisms (linear bijections): $\mathcal{F}_a \leftarrow \mathcal{F}(D_0)$; $\mathcal{F}(D_{i-1}) \to \mathcal{F}(D_i \cap D_{i-1}) \leftarrow \mathcal{F}(D_i)$ for $i = 1,\dots,N$; and $\mathcal{F}(D_N) \to \mathcal{F}_b$. Composing all these maps, we get an isomorphism $s \mapsto s^\gamma$ from $\mathcal{F}_a$ to $\mathcal{F}_b$. We know that this isomorphism depends only on the homotopy class of γ in $\Pi_1(\Omega; a, b)$. In case $\mathcal{F}$ is $\mathcal{F}_{\underline{a}}$ or $\mathcal{F}_A$, this follows from the principle of monodromy. For an arbitrary local system, this can be proved in exactly the same way (see the book of Deligne). To summarize, we get a map:

$$\Pi_1(\Omega; a, b) \to \mathrm{Iso}(\mathcal{F}_a, \mathcal{F}_b).$$

Last, we know that the composition of pathes (or homotopy classes) gives rise to the composition of these maps: the proof is just as easy in the case of an arbitrary local system as in the case of $\mathcal{F}_{\underline{a}}$ or $\mathcal{F}_A$. Therefore, taking $a = b$, we get an anti-morphism of groups:

$$\rho_{\mathcal{F},a} : \pi_1(\Omega; a) \to \mathrm{GL}(\mathcal{F}_a).$$

This is the monodromy representation of the local system $\mathcal{F}$ at the base point a.

Remark 7.5.1 Call "opposite" of a group G with multiplication $x * y$ the group G° having the same elements but the group law $x \star y := y * x$. Then an anti-morphism of groups $G \to H$ is the same thing as a morphism $G^\circ \to H$. For this reason, it is sometimes said that the monodromy representation is:

$$\big(\pi_1(\Omega; a)\big)^\circ \to \mathrm{GL}(\mathcal{F}_{\underline{a}}).$$

However, the distinction will have no consequence for us.

Definition 7.5.2 The *monodromy group* of $\mathcal{F}$ at the base point a is the image of the monodromy representation:

$$\mathrm{Mon}(\mathcal{F}, a) := \mathrm{Im}\rho_{\mathcal{F},a} \subset \mathrm{GL}(\mathcal{F}_a).$$

In the case of $\mathcal{F}_{\underline{a}}$ and $\mathcal{F}_A$, writing z_0 the base point, we will denote $\rho_{\underline{a},z_0}$ or ρ_{A,z_0} the monodromy representation and $\mathrm{Mon}(E_{\underline{a}}, z_0)$ or $\mathrm{Mon}(S_A, z_0)$ the monodromy group.

Now we illustrate by an easy result the reason one can consider monodromy theory as a "galoisian" theory. Remember that the map $\mathcal{F}(\Omega) \mapsto \mathcal{F}_a$ is injective, so that one can identify a "global section" $s \in \mathcal{F}(\Omega)$ with its germ s_a at a: global sections are just germs which can be extended all over Ω.

Theorem 7.5.3 *The germ $s \in \mathcal{F}_a$ is fixed by the monodromy action if, and only if, $s \in \mathcal{F}(\Omega)$.*

Proof. - The hypothesis means: $\forall g \in \pi_1(\Omega, a)$, $\rho_{\mathcal{F},a}(g)(s) = s$. In this case, it translates to: for every loop γ based at a, one has $s^\gamma = s$. This implies that, for every point $b \in \Omega$ and every path γ from a to b, the section $s^\gamma \in \mathcal{F}_b$ depends on b alone and not on the path γ. Then one can glue together all these germs to make a section $s \in \mathcal{F}(\Omega)$ whose germ at a is the given one. (We already met a similar argument in section 5.2.) □

Remark 7.5.4 The similarity with Galois theory of algebraic equations is as follows. Suppose $P(x) = 0$ is an irreducible equation over $\mathbf{Q}$. Then, a rational expression $A(x_1, \ldots, x_n)/B(x_1, \ldots, x_n)$ in the roots x_i of P is a rational number if, and only if, it is left invariant by all permutations of the roots. Here, the field K of all rational expressions of the roots plays the role of $\mathcal{F}_a$; the base field $\mathbf{Q}$ plays the role of $\mathcal{F}(\Omega)$; the symmetric group plays the role of the fundamental group; and the Galois group, which is the image of the symmetric group, plays the role of the monodromy group. One good source to pursue the analogy further is the book by Adrien and Régine Douady, "Algèbre et théories galoisiennes".

Corollary 7.5.5 *(i) If the monodromy group of S_A is trivial, then S_A admits a fundamental matricial solution on Ω.*
(ii) If the monodromy group of $E_{\underline{a}}$ is trivial, then $E_{\underline{a}}$ admits a fundamental system of solutions on Ω.

Corollary 7.5.6 *(i) For every simply connected domain U of Ω, S_A admits a fundamental matricial solution on U.*
(ii) For every simply connected domain U of Ω, $E_{\underline{a}}$ admits a fundamental system of solutions on U.

Using the last of the "basic properties" of section 3.1, one obtains:

Corollary 7.5.7 *(i) The r.o.c. of every fundamental matricial solution of S_A at z_0 is $\geq d(z_0, \partial\Omega)$.*
(ii) The r.o.c. of every fundamental system of solutions of $E_{\underline{a}}$ at z_0 is $\geq d(z_0, \partial\Omega)$.

Proof. - Indeed, they can be extended to holomorphic functions on the corresponding disk (since this disk is simply connected). □

Exercice 7.5.8 (i) Why is the map $\mathcal{F}(\Omega) \mapsto \mathcal{F}_a$ injective ?
(ii) Prove rigorously the glueing argument in the theorem.

Dependency on the base point a. If $a, b \in \Omega$, since Ω is arcwise connected (as any domain in **C**), there is a path γ from a to b. This path induces an isomorphism $\phi_\gamma : \pi_1(\Omega; a) \to \pi_1(\Omega; b)$, $[\lambda] \mapsto [\gamma^{-1}.\lambda.\gamma]$. Actually, ϕ_γ only depends on the homotopy class $[\gamma]$ of γ, so we could as well denote it $\phi_{[\gamma]}$, but this does not matter. Therefore, all the fundamental groups of Ω are isomorphic, but beware that the isomorphisms are not canonical:

Exercice 7.5.9 Let γ' be another path from a to b. Then $\phi_{\gamma'}^{-1} \circ \phi_\gamma$ is an automorphism of the group $\pi_1(\Omega; a)$ and $\phi_\gamma \circ \phi_{\gamma'}^{-1}$ is an automorphism of the group $\pi_1(\Omega; b)$. Show that these are inner automorphisms (that is, of the form $g \mapsto g_0 g g_0^{-1}$).

The path γ also induces an isomorphism $u_\gamma : \mathcal{F}_a \to \mathcal{F}_b$, $s \mapsto s^\gamma$, whence, by conjugation, an isomorphism $\psi_\gamma : \mathrm{GL}(\mathcal{F}_a) \to \mathrm{GL}(\mathcal{F}_b)$, $u \mapsto u_\gamma \circ u \circ u_\gamma^{-1}$. (Again, these isomorphisms only depend on the homotopy class $[\gamma]$.) This gives a commutative diagram:

$$\begin{tikzcd} \pi_1(\Omega; a) \arrow[r, "\rho_a"] \arrow[d, "\phi_\gamma"'] & \mathrm{GL}(\mathcal{F}_a) \arrow[d, "\psi_\gamma"] \\ \pi_1(\Omega; b) \arrow[r, "\rho_b"'] & \mathrm{GL}(\mathcal{F}_b) \end{tikzcd}$$

Indeed, the element $[\lambda] \in \pi_1(\Omega; a)$ goes down to $[\gamma^{-1}.\lambda.\gamma]$, then right to $u_\gamma \circ u_\lambda \circ u_\gamma^{-1}$; and it goes right to u_λ, then down to $u_\gamma \circ u_\lambda \circ u_\gamma^{-1}$.

As a consequence, ψ_γ sends $\mathrm{Mon}(\mathcal{F}, a) = \mathrm{Im}\rho_a$ to $\mathrm{Mon}(\mathcal{F}, b) = \mathrm{Im}\rho_b$: the monodromy groups of $\mathcal{F}$ at different points are all isomorphic (but the isomorphisms are not canonical).

Matricial monodromy representation. Let $\mathcal{B}$ be a basis of $\mathcal{F}_a$, whence an isomorphism $\mathbf{C}^n \to \mathcal{F}_a$, $X_0 \mapsto \mathcal{B}X_0$. The automorphism u_λ of analytic continuation along a loop λ based at a transforms $\mathcal{B}$ into a new base $\mathcal{B}^\lambda = \mathcal{B}M_{[\lambda]}$, where $M_{[\lambda]} \in \mathrm{GL}_n(\mathbf{C})$. It transforms an element $X = \mathcal{B}X_0 \in \mathcal{F}_a$ into $X^\lambda = \mathcal{B}^\lambda X_0 = \mathcal{B}(M_{[\lambda]}X_0) \in \mathcal{F}_a$. Therefore, in the space $\mathbf{C}^n$, the automorphism of analytic continuation is represented by the linear map $X_0 \mapsto M_{[\lambda]}X_0$. In this way, our monodromy representation has been conjugated to a *matricial monodromy representation* $\pi_1(\Omega, a) \to \mathrm{GL}_n(\mathbf{C})$, $[\lambda] \mapsto M_{[\lambda]}$. Note that, as usual, this is an anti-morphism of groups. Its image is the *matricial monodromy group* of $\mathcal{F}$ at a with respect to the basis $\mathcal{B}$. So let $\mathcal{B}' = \mathcal{B}P$ be another basis, with $P \in \mathrm{GL}_n(\mathbf{C})$. The corresponding monodromy matrices are defined by the relations $\mathcal{B}'^\lambda = \mathcal{B}'M'_{[\lambda]}$, where $M'_{[\lambda]} \in \mathrm{GL}_n(\mathbf{C})$. On the other hand:

$$\mathcal{B}'^\lambda = (\mathcal{B}P)^\lambda = \mathcal{B}^\lambda P = \mathcal{B}M_{[\lambda]}P = \mathcal{B}'P^{-1}M_{[\lambda]}P \text{ so that } M'_{[\lambda]} = P^{-1}M_{[\lambda]}P.$$

Therefore, changing the base yields a conjugated representation: the matricial monodromy groups of $\mathcal{F}$ at a with respect to various bases are all conjugated as subgroups of $\mathrm{GL}_n(\mathbf{C})$.

In the case of the sheaf of solutions of a differential system or equation, the "initial condition map" $\mathcal{F}_{A\,z_0} \to \mathbf{C}^n$, $X \mapsto X(z_0)$ or $\mathcal{F}_{\underline{a}\,z_0} \to \mathbf{C}^n$, $f \mapsto (f(z_0), \dots, f^{(n-1)}(z_0))$ allows us to make a canonical choice of a basis: indeed, we can use that one whose image by the initial condition map is the canonical basis of $\mathbf{C}^n$. For instance, in the case of systems, that means that $\mathcal{B}$ has as elements the columns of the fundamental matricial solution such that $\mathcal{X}(z_0) = I_n$. However, this natural choice is not always the best do describe the matricial monodromy group:

Example 7.5.10 Consider the equation $z^2 f'' - zf' + f = 0 \Leftrightarrow f'' - z^{-1}f' + z^{-2}f = 0$ on $\Omega := \mathbf{C}^*$, take the base point $z_0 := 1$ and the usual fundamental loop λ such that $I(0, \lambda) = +1$. For the obvious basis $\mathcal{B} := (z, z\log z)$, one has $\mathcal{B}^\lambda = (z, z(\log z + 2i\pi)) = \mathcal{B}M_{[\lambda]}$ with $M_{[\lambda]} = \begin{pmatrix} 1 & 2i\pi \\ 0 & 1 \end{pmatrix}$. However, the basis which corresponds to the "canonical" initial conditions $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is rather $\mathcal{B}' = (z - z\log z, z\log z)$, that is BP with $P := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. The corresponding monodromy matrix is $M'_{[\lambda]} = P^{-1}M_{[\lambda]}P = \begin{pmatrix} 1 - 2i\pi & 2i\pi \\ -2i\pi & 1 + 2i\pi \end{pmatrix}$. It is clearly easier to compute the matricial monodromy group generated by $M_{[\lambda]}$ than the matricial monodromy group generated by $M'_{[\lambda]}$.

Exercice 7.5.11 Check that $\mathcal{B}'^\lambda = \mathcal{B}'M'_{[\lambda]}$ in the example.

7.6 Holomorphic and meromorphic equivalences of systems

Gauge transformations and equivalence of systems. The basic idea here is that of change of unknown function. Instead of studying (or trying to solve) the system $X' = AX$, we introduce $Y := FX$, where F is an invertible matrix of functions. We then find that:

$$Y' = (FX)' = F'X + FX' = F'X + FAX = (F' + FA)F^{-1}Y,$$

i.e. $Y' = BY$ where $B := F'F^{-1} + FAF^{-1}$. Conversely, since F is invertible, one can see that, if $Y' = BY$, then $X := F^{-1}Y$ satisfies $X' = AX$. The matrix F as well as the map $X \mapsto FX$ are called a *gauge transformation*. Note that there is no corresponding notion for scalar equations.

Definition 7.6.1 We shall write $F[A] := F'F^{-1} + FAF^{-1}$. If $B = F[A]$, we shall write $F : A \simeq B$ or $F : S_A \simeq S_B$, or even[4] $F : A \to B$ or $F : S_A \to S_B$.
If $F \in \mathrm{GL}_n(\mathcal{O}(\Omega))$, we shall say that the systems S_A and S_B on Ω are *holomorphically equivalent*, or *holomorphically isomorphic*, and we write $A \underset{h}{\sim} B$, or $S_A \underset{h}{\sim} S_B$.
If $F \in \mathrm{GL}_n(\mathcal{M}(\Omega))$, we shall say that the systems S_A and S_B on Ω are *meromorphically equivalent*, or *meromorphically isomorphic*, and we write $A \underset{m}{\sim} B$, or $S_A \underset{m}{\sim} S_B$.

Example 7.6.2 The relation $A = F[0_n]$ means that F is a fundamental matricial solution of S_A. Thus, $A \underset{h}{\sim} 0_n$, resp. $A \underset{m}{\sim} 0_n$, means that A admits a fundamental matricial solution that is holomorphic, resp. meromorphic on Ω.

Exercice 7.6.3 Give a necessary and sufficient condition for two rank one systems $x' = ax$ and $y' = by$ to be holomorphically, resp. meromorphically equivalent.

Proposition 7.6.4 *One has the following equalities:* $I_n[A] = A$ *and* $G[F[A]] = (GF)[A]$; *and the following logical equivalence:* $B = F[A] \Leftrightarrow A = F^{-1}[B]$.

Proof. - Easy computations, left to the reader ! □

Corollary 7.6.5 *Holomorphic and meromorphic equivalences are indeed equivalence relations.*

Remark 7.6.6 One can say that the group $\mathrm{GL}_n(\mathcal{O}(\Omega))$ operates on $\mathrm{Mat}_n(\mathcal{O}(\Omega))$, but the similar statement for $\mathrm{GL}_n((\Omega))$ is not correct: the operation is partial, since $F[A]$ could have poles.

The following result says that, in essence, one does not really generalize the theory by studying systems rather than scalar differential equations.

Theorem 7.6.7 (Cyclic vector lemma) *Every system S_A is meromorphically equivalent to a system coming from a scalar equation $E_{\underline{a}}$, i.e. $S_{A_{\underline{a}}}$.*

Proof. - We want to find $F \in \mathrm{GL}_n(\mathcal{M}(\Omega))$ such that $F' + FA = A_{\underline{a}}F$ for some functions $a_1, \ldots, a_n$. We shall actually find such an F with holomorphic coefficients. To ensure that $F \in \mathrm{GL}_n(\mathcal{M}(\Omega))$, it will then be sufficient to ensure that $(\det F)(z_0) \neq 0$ at one arbitrary point $z_0 \in \Omega$. Obviously, we can as well assume that $0 \in \Omega$ and choose $z_0 := 0$. Call $L_1, \ldots, L_n$ the lines of F. Then:

$$F' + FA = A_{\underline{a}}F \Longleftrightarrow L_1' + L_1A = L_2, \ldots, L_{n-1}' + L_{n-1}A = L_n \text{ and } L_n' + L_nA = -(a_nL_1 + \cdots + a_1L_n).$$

Since $a_1, \ldots, a_n$ are not imposed, it is therefore enough to choose L_1 holomorphic on Ω and such that the sequence defined by $L_{i+1} := L_i' + L_iA$ for $i = 1, \ldots, n-1$ produces a basis $(L_1, \ldots, L_n)$ of $\mathcal{M}(\Omega)^n$. (Such a vector L_1 is called a *cyclic vector*, whence the name of the theorem.) To that end, we shall simply require that $\det(L_1, \ldots, L_n)(0) = 1$.
Now, choose a fundamental matricial solution $\mathcal{X}$ at 0 such that $\mathcal{X}(0) = I_n$. Putting $M_i := L_i\mathcal{X}$ (recall that these are line vectors), one sees that $L_{i+1} := L_i' + L_iA \Leftrightarrow M_{i+1} = M_i'$. Thus, we must choose $M_i := M_1^{(i-1)}$ for $i = 2, \ldots, n$. On the other hand, $\det(M_1, \ldots, M_n)(0) = \det(L_1, \ldots, L_n)(0)$ since

[4] This is the notation for the more general notion of "morphism" (see remark at the end of the chapter), so in this case we would say explicitly "the equivalence $F : A \to B$" or "the isomorphism $F : A \to B$".

$\det \mathcal{X}(0) = 1$. Therefore, we look for M_1 such that $\det(M_1, \dots, M_1^{(n-1)})(0) = 1$. This is very easy: calling $E_1, \dots, E_n$ the canonical basis of $\mathbf{C}^n$, we could just take $M_1 = E_1 + zE_2 + \cdots + \frac{z^{n-1}}{(n-1)!} E_n$. However, in that case, $L_1 := M_1 \mathcal{X}^{-1}$ would not be holomorphic on the whole of Ω. Therefore we *truncate* the vector $M_1 \mathcal{X}^{-1}$ (which is a power series at 0) in order to eliminate terms containing z^n. This does not change the condition on the first $(n-1)$ derivatives at 0 of M_1.
To summarize: L_1 is the truncation of the line vector:

$$\left(E_1 + zE_2 + \cdots + \frac{z^{n-1}}{(n-1)!} E_n \right) \mathcal{X}^{-1}$$

up to degree $(n-1)$; the line vectors $L_2, \dots, L_n$ are defined by the recursive formulas $L_{i+1} := L_i' + L_i A$ for $i = 1, \dots, n-1$; the gauge ransformation matrix F has lines $L_2, \dots, L_n$; then $F[A] = A_{\underline{a}}$ for some $\underline{a}$. $\square$

Remark 7.6.8 We cannot conclude that S_A is holomorphically equivalent to $S_{A_{\underline{a}}}$, because F is holomorphic on Ω but F^{-1} might have poles if $\det F$ has zeroes. A more serious drawback of the theorem is that the a_i are meromorphic on Ω and not necessarily holomorphic. There is no corresponding result that guarantees an equation with holomorphic coefficients.

The problem of finding cyclic vectors is a practical one and software dedicated to formal treatment of differential equations uses more efficient algorithms than the one shown above, which requires finding a fundamental matricial solution. Note however that taking for L_1 a vector of polynomials of degree $(n-1)$ *at random* will yield a cyclic vector with probability 1 ! (Suggestion: prove it.)

Example 7.6.9 Let $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathcal{O}(\Omega)$. Take $L_1 := (1, 0)$. Then $L_2 := L_1' + L_1 A = (a, b)$, so that (L_1, L_2) is a basis except if $b = 0$, which is obviously an exceptionnal condition. In the same way, $(0, 1)$ is a cyclic vector, except if $c = 0$.

Exercice 7.6.10 Starting from an arbitrary system of rank 2, find an equivalent scalar equation. (Method: by brute force it is not very difficult.)

Meromorphic equivalence and sheaves of solutions. We now describe more precisely the effect of a meromorphic gauge transformation on solutions. Let $F : A \simeq B$ be such a transformation. Let $X \in \mathcal{F}_A(U)$ be a solution of S_A on U. Then, FX is a meromorphic solution of $\mathcal{F}_B$ on U. But we know from corollary 7.3.4 that then FX is analytic: $FX \in \mathcal{F}_B(U)$. Thus, $X \mapsto FX$ is a linear map $\mathcal{F}_A(U) \to \mathcal{F}_B(U)$. For the same reason, $Y \mapsto F^{-1}Y$ is a linear map $\mathcal{F}_B(U) \to \mathcal{F}_A(U)$, and it is the inverse of the previous one, so that they are isomorphisms.

Definition 7.6.11 An isomorphism $\phi : \mathcal{F} \to \mathcal{F}'$ of sheaves of complex linear spaces on Ω is a family (ϕ_U) indexed by the open subsets U of Ω, where each $\phi_U : \mathcal{F}(U) \to \mathcal{F}'(U)$ is an isomorphism (of complex linear spaces) and where the family is compatible with the restriction maps in the following sense: if $V \subset U$, then the following diagram is commutative:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi_U"] \arrow[d] & \mathcal{F}'(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r, "\phi_V"'] & \mathcal{F}'(V) \end{tikzcd}$$ (the vertical maps are the restriction maps).

Example 7.6.12 If $A = A_{\underline{a}}$, the maps $\mathcal{F}_{\underline{a}}(U) \to \mathcal{F}_A(U)$ defined in proposition 7.2.3 make up an isomorphism of sheaves.

Theorem 7.6.13 *The systems S_A and S_B are meromorphically equivalent if, and only if the sheaves $\mathcal{F}_A$ and $\mathcal{F}_B$ are isomorphic.*

Proof. - We just proved that $S_A \underset{h}{\sim} S_B$ implies $\mathcal{F}_A \simeq \mathcal{F}_B$. (The compatibility with restriction maps was obviously satified.) Assume conversely that (ϕ_U) is an isomorphism from $\mathcal{F}_A$ to $\mathcal{F}_B$. From the compatibility with restriction maps, it follows that (ϕ_U) induces isomorphisms $\phi_{z_0} : \mathcal{F}_{A,z_0} \to \mathcal{F}_{B,z_0}$ between the spaces of germs at an arbitrary $z_0 \in \Omega$. As a consequence, a fundamental matricial solution $\mathcal{X}$ of S_A at z_0 has as an image a fundamental matricial solution $\mathcal{Y} := \phi_{z_0}(\mathcal{X})$ of S_B at z_0.
Again from the compatibility conditions (and from the fact that $\mathcal{F}_A$, $\mathcal{F}_B$ are local systems), it follows that, along a path γ from z_0 to z_1, endowed with the usual small disks $D_0, \ldots, D_N$, the successive continuations $\mathcal{X}_0 := \mathcal{X}, \ldots, \mathcal{X}_N$ and $Y_0 := \mathcal{Y}, \ldots, \mathcal{Y}_N$ satisfy: $\phi_{D_i}(\mathcal{X}_i) = \mathcal{Y}_i$. In other words, $\phi_{z_1}(\mathcal{X}^\gamma) = \mathcal{Y}^\gamma$. It is in particular true that, for any loop λ based at z_0, one has $\phi_{z_0}(\mathcal{X}^\lambda) = \mathcal{Y}^\lambda$.
On the other hand, $\mathcal{X}^\lambda = \mathcal{X} M_{[\lambda]}$ and $\mathcal{Y}^\lambda = \mathcal{Y} N_{[\lambda]}$ (the monodromy matrices) and, by linearity of ϕ_{z_0}, one has:

$$\mathcal{Y} N_{[\lambda]} = \mathcal{Y}^\lambda = \phi_{z_0}(\mathcal{X}^\lambda) = \phi_{z_0}(\mathcal{X} M_{[\lambda]}) = \phi_{z_0}(\mathcal{X}) M_{[\lambda]} = \mathcal{Y} M_{[\lambda]},$$

i.e. $M_{[\lambda]} = N_{[\lambda]}$.
Now, if we set $F := \mathcal{Y}\mathcal{X}^{-1} \in \mathrm{GL}_n(\mathcal{M}_{z_0})$ (the meromorphic germs at z_0), we see that:

$$F^\lambda = \mathcal{Y}^\lambda (\mathcal{X}^\lambda)^{-1} = (\mathcal{Y} M_{[\lambda]})(\mathcal{X} M_{[\lambda]})^{-1} = \mathcal{Y}\mathcal{X}^{-1} = F.$$

As in the "galoisian" theorem 7.5.3, we conclude that $F \in \mathrm{GL}_n(\mathcal{M}(\Omega))$.
Last, from the facts that $\mathcal{X}' = A\mathcal{X}$ and $(F\mathcal{X})' = B(F\mathcal{X})$, it follows that $F' + FA = BF$ (we use the fact that $\mathcal{X}$ is invertible). $\Box$

Note that it was made an essential use of monodromy considerations. The information carried by the local systems $\mathcal{F}_A$, $\mathcal{F}_B$ is clearly topological in nature.

Meromorphic equivalence and monodromy representations. Again consider a meromorphic isomorphism $F : A \to B$. Fix $z_0 \in \Omega$ and fundamental matricial solutions $\mathcal{X}$, $\mathcal{Y}$ of S_A, S_B at z_0. (We do not require that $\mathcal{Y} = F\mathcal{X}$.) The monodromy representations $\pi_1(\Omega, z_0) \to \mathrm{GL}(\mathcal{F}_{A,z_0})$, $[\lambda] \mapsto M_{[\lambda]}$ and $\pi_1(\Omega, z_0) \to \mathrm{GL}(\mathcal{F}_{B,z_0})$, $[\lambda] \mapsto N_{[\lambda]}$ are characterized by the relations $\mathcal{X}^\lambda = \mathcal{X} M_{[\lambda]}$ and $\mathcal{Y}^\lambda = \mathcal{Y} N_{[\lambda]}$.

Since $F\mathcal{X}$ is an invertible matrix and a matricial solution of S_B at z_0 (because of the relation $F' = BF - FA$), it can be written $F\mathcal{X} = \mathcal{Y} P$, where $P \in \mathrm{GL}_n(\mathbf{C})$. Since F is globally defined on Ω, it is fixed by the monodromy: $F^\lambda = F$ for any loop λ based at z_0. The effect of λ on the equality $F\mathcal{X} = \mathcal{Y}P$ is $(F\mathcal{X})^\lambda = (\mathcal{Y}P)^\lambda$, whence:

$$F^\lambda \mathcal{X}^\lambda = \mathcal{Y}^\lambda P^\lambda \Longrightarrow F\mathcal{X}^\lambda = \mathcal{Y}^\lambda P \Longrightarrow F\mathcal{X}M_{[\lambda]} = \mathcal{Y}N_{[\lambda]}P \Longrightarrow \mathcal{Y}PM_{[\lambda]} = \mathcal{Y}N_{[\lambda]}P \Longrightarrow PM_{[\lambda]} = N_{[\lambda]}P.$$

Definition 7.6.14 Two linear representations $G \underset{\rho}{\rightarrow} \mathrm{GL}(E)$ and $G \underset{\rho'}{\rightarrow} \mathrm{GL}(E')$ are said to be *equivalent* or *conjugate* or *isomorphic* if there exists an isomorphism $p : E \rightarrow E'$ such that:

$$\forall g \in G\,,\ \rho'(g) \circ p = p \circ \rho(g) \quad i.e. \quad \rho'(g) = p \circ \rho(g) \circ p^{-1}.$$

Example 7.6.15 All the monodromy representations attached to S_A at the point z_0 are isomorphic. This includes the intrinsic representation $\pi_1(\Omega, z_0) \rightarrow \mathrm{GL}(\mathcal{F}_{A,z_0})$ as well as all the matricial representations arising from the choice of a basis of $\mathcal{F}_{A,z_0}$.

Theorem 7.6.16 *Two systems are meromorphically equivalent if, and only if, their monodromy representations at some arbitrary point are isomorphic.*

Proof. - We have already proved that, if $S_A \underset{m}{\sim} S_B$, then their monodromy representations at z_0 are conjugate.
So we assume conversely that these representations are conjugate, *i.e.* there exists $P \in \mathrm{GL}_n(\mathbf{C})$ such that, for all loops λ based at z_0, one has $PM_{[\lambda]} = N_{[\lambda]}P$ (where these monodromy matrices come from $\mathcal{X}$, $\mathcal{Y}$ and z_0 chosen as before).
We then put $F := \mathcal{Y}P\mathcal{X}^{-1}$. This is a meromorphic germ of invertible matrix and, since $F\mathcal{X} = \mathcal{Y}P$ is a matricial solution of S_B, the usual computation implies $F' + FA = BF$. On the other hand, the effect of monodromy on F is as follows:

$$F^\lambda = (\mathcal{Y}P\mathcal{X}^{-1})^\lambda = \mathcal{Y}^\lambda P^\lambda (\mathcal{X}^\lambda)^{-1} = \mathcal{Y}N_{[\lambda]}PM_{[\lambda]}^{-1}\mathcal{X}^{-1} = \mathcal{Y}P\mathcal{X}^{-1} = F.$$

Thus, by the usual "galoisian" argument, F is global: $F \in \mathrm{GL}_n(\mathcal{M}(\Omega)$ and $S_A \underset{m}{\sim} S_B$. □

Corollary 7.6.17 *The monodromy group* $Mon(\mathcal{F}_A, z_0)$ *is trivial if, and only if the system* $\mathcal{F}_A$ *admits a meromorphic fundamental matricial solution.*

Proof. - Indeed, the second statement is equivalent to being meromorphically equivalent to 0_n. □

Exercice 7.6.18 The construction of the monodromy representation really depends only on the local system $\mathcal{F}$ (not the fact that it comes from a differential system). Prove that two local systems are isomorphic (as sheaves) if, and only if, the attached monodromy representations are equivalent.

Remark 7.6.19 In all three definitions of isomorphism (meromorphic equivalence of differential systems, isomorphism of sheaves, conjugacy of representations), one can relax the requirement of bijectivity. One thereby obtains the notion of *morphism* (of differential systems, of sheaves, of representations) and one proves more generally that (meromorphic) morphisms from S_A to and S_B, morphisms from $\mathcal{F}_A$ to $\mathcal{F}_B$ and morphisms from the monodromy representation $\pi_1(\Omega, z_0) \rightarrow \mathrm{GL}(\mathcal{F}_{A,z_0})$ to the monodromy representation $\pi_1(\Omega, z_0) \rightarrow \mathrm{GL}(\mathcal{F}_{B,z_0})$ correspond bijectively to each other. See the book of Deligne.

Chapter 8

Regular singular points and the local Riemann-Hilbert correspondance

References for this chapter are chapter 8.3 of the book by Ahlfors (for a gentle introduction to the problem); and the following, mostly about technical information on scalar differential equations:

- Birkhoff and Rota, "Ordinary differential equations", chapter 9.
- Coddington and Levinson, "Theory of ordinary differential equations", chapter 4 (4.5 to 4.8).
- Hille, "Ordinary differential equations in the complex domain", chapter 5 (5.1 to 5.3).
- Ince, "Ordinary differential equations", chapter 16.

The book by Ince, although somewhat old fashioned, is a great classic and rich in information.

8.1 Introduction and motivation

Many special functions discovered in mathematics and in physics since the XVIIIthcentury were found to be solutions of linear differential equations with polynomial coefficients. This includes the exponential and logarithm functions, the functions z^α and the hypergeometric series, which we shall study in detail in chapter 9. Note that this does not include the most famous Gamma, Zeta and Theta functions: but these ones do satisfy other kinds of "functional equations" that also proved useful in their study.

Exercice 8.1.1 Prove that $1/\log$ is not the solution of a linear differential equation with polynomial coefficients.

We want to make a *global* study of such functions. It turned out (as experience in the domain accumulated) that their global behaviour is extremely dependent on the singularities of the equation, that is the singularities of the coefficients a_i when the equation is written in the form $E_{\underline{a}}$. So it is worth starting with a *local* study near the singularities. The most important features of a solution near a singularity are:

1. Its monodromy. This says how far the solution is of being uniform. Solutions "ramify" and for this reason singularities are sometimes called "branch points".

2. Its rate of growth (or decay), which maybe moderate or have the physical character of an explosion.

That monodromy alone is not a sufficient feature to characterize solutions near a singularity is shown by the following examples: $zf' - f = 0$, $zf' + f = 0$ and $z^2 f' + f = 0$. The first one has z as a basis of solutions: the singularity is only apparent. The second one has $1/z$ as a basis of solutions: the solutions have a simple pole. The third one has $e^{1/z}$ as a basis of solutions: the solutions may have exponential growth or decay, or even spiralling, according to the direction in approaching the singularity 0. Yet, in all three cases, the monodromy is trivial.

So the first step of the study (beyond ordinary points where Cauchy theorem applies) is the case of a singularity where all solutions have moderate growth. For uniform solutions near 0, this excludes $e^{1/z}$ (and actually all solutions having an essential singularity at 0) but includes all meromorphic solutions, since the condition $f(z) = O(z^{-N})$ (see the corollary of theorem 3.2.1) means that they have polynomial growth as functions of $1/z$. However, for "multivalued" solutions (*i.e.* those with non trivial monodromy), the definition has to be adapted:

Example 8.1.2 The analytic continuation L of the log function along the infinite path $\gamma(t) := e^{-t+\mathrm{i}e^{t^2}}$, $t \in \mathbf{R}_+$, takes the value $L(\gamma(t)) = -t + \mathrm{i}e^{t^2}$, so that $|L(\gamma(t))| \geq e^{t^2}$, while $|\gamma(t)| = e^{-t}$: clearly the condition $L(z) = O(z^{-N})$ is satisfied for no N along this path. (This example will be detailed in section 8.2.)

Exercice 8.1.3 Find a similar example with a function z^α.

So we shall give in section 8.2 a reasonable definition of moderate growth (one which would not exclude the logarithm and z^α functions). We shall then find that equations all of whose solutions have moderate growth can be classified by their monodromy: this is the local Riemann-Hilbert correspondance[1]. Of course, the equivalence relation used (through gauge transformations) must be adapted so as to take in account what goes on at 0: there will be a new definition of meromorphic equivalence.

Conventions for this chapter. All systems and equations considered will have coefficients which are holomorphic in some puntured disk $\dot{D} := \overset{\circ}{\mathrm{D}}(0,R) \setminus \{0\}$, and which are meromorphic at 0; in other words, they will be elements of the field $\mathcal{M}_0 = \mathbf{C}(\{z\})$ (meromorphic germs at 0). This means that we consider only the singularity 0: other points will be treated in examples. (One can always change variables so as to reduce the problem to this case.)

Definition 8.1.4 Let $A, B \in \mathrm{Mat}_n(\mathbf{C}(\{z\}))$. We say that A and B, or S_A and S_B are *meromorphically equivalent at* 0 if there exists $F \in \mathrm{GL}_n(\mathbf{C}(\{z\}))$ such that $F[A] = B$, *i.e.* $F' = BF - FA$. We then write $A \sim B$ or $S_A \sim S_B$.

Note that, the zeroes and poles of a non trivial meromorphic function being isolated, A and B are then holomorphically equivalent in some punctured neighborhood of 0.

Exercice 8.1.5 Are the equations $zf' - f = 0$, $zf' + f = 0$ and $z^2 f' + f = 0$ meromorphically equivalent at 0 ?

[1]The global Riemann-Hilbert correspondance will be studied in chapters 9 and 10.

8.2 The condition of moderate growth in sectors

To understand the condition we are going to use, let us examine more closely the example of the logarithm in the previous section.

Example 8.2.1 Let $\gamma(t) := e^{-t+\mathrm{i}e^{t^2}}$, for $t \geq 0$. Then $\gamma(t) \to 0$ as $t \to +\infty$. This infinite path starts at $\gamma(0) = e^{\mathrm{i}} \in \mathbf{C} \setminus \mathbf{R}_-$, the domain of the principal determination of the logarithm, and $\log \gamma(t) = -t + \mathrm{i}e^{t^2}$ as long as the path does not leave the domain, that is $t < \sqrt{\ln \pi}$.
To give a proper meaning to analytic continuation, we "thicken" the image curve (which is a spiral) into a simply connected domain $U \supset \mathrm{Im}\gamma$ such that $0 \in \overline{U}$. Let

$$V := \{(t,u) \in \mathbf{R}^2 \mid t \geq 0 \text{ and } |u| \leq \operatorname{argsh}(\pi e^{-t^2})\}.$$

The latter condition implies that:

$$(t,u_1),(t,u_2) \in V \Rightarrow \left| e^{u_2+t^2} - e^{u_1+t^2} \right| < 2\pi$$

and allows us to deduce that $(t,u) \mapsto e^{-t+\mathrm{i}e^{t^2+u}}$ is a homeomorphism from V to a "thick spiral" $U \supset \mathrm{Im}\gamma$ (the latter curve being the image of the subset $u = 0$ of V). Since V is simply connected, V is also. Now, by a (by now) standard continuity argument, the determination of the logarithm on the simply connected domain U which coincides with log on their common domain is given by the formula $L\left(e^{-t+\mathrm{i}e^{t^2+u}}\right) = -t + \mathrm{i}e^{t^2+u}$; in particular, for $t \in \mathbf{R}_+$, it takes the value $L(\gamma(t)) = -t + \mathrm{i}e^{t^2}$, so that (as we already saw) $|L(\gamma(t))| \geq e^{t^2}$, while $|\gamma(t)| = e^{-t}$: clearly the condition $L(z) = O(z^{-N})$ is not satisfied in U for any N.

We now consider a punctured disk $\dot{D} := \mathring{\mathrm{D}}(0,R) \setminus \{0\}$ and an analytic germ f at some point $a \in \dot{D}$ such that f admits analytic continuation along all pathes in $\dot{D}$ originating in a. (This will be the case for solutions of linear differential equations.) The collection F of all germs obtained from f in this way is called a *multivalued function* on $\dot{D}$. For any open subset $U \subset \dot{D}$, an analytic function on U all of whose germs belong to the collection F is called a *determination of F on U*. (Such a determination is therefore an element of $O(U)$.) For simply connected domains, such determinations always exist. For an arbitrary domain U, determinations subject to initial conditions are unique, *i.e.* two determinations of F on U which take the same value w_0 at some $z_0 \in U$ are equal. Clearly, one can linearly combine, multiply and derive multivalued functions: they form a differential algebra containing $O(\dot{D})$.

In the following definition, we shall use the following notation for open angular sectors with vertex at 0; if $0 < b - a < 2\pi$, then:

$$S_{a,b} := \{re^{\mathrm{i}\theta} \mid r > 0 \text{ and } a < \theta < b\}.$$

Definition 8.2.2 We say that a multivalued function F on $\dot{D}$ has *moderate growth in sectors* if, for any $a,b \in \mathbf{R}$ such that $0 < b - a < 2\pi$ and for any determination f of F on the simply connected domain $U := \dot{D} \cap S_{a,b}$, there exists $N \in \mathbf{N}$ such that $f(z) = O(z^{-N})$ as $z \to 0$ in U. (The exponent N may depend on the sector and on the determination.)

Note that restricting to a smaller punctured disk (with radius $R' < R$) does not affect the condition (or its negation), so, in practice, we expect it to be true (or false) for some unspecified R and do not usually mention the punctured disk $\dot{D}$.

Example 8.2.3 Any determination of the logarithm on $\dot{D} \cap S_{a,b}$ is such that $L(re^{i\theta}) = \ln r + i(\theta + 2k_0\pi)$ for $r > 0$ and $a < \theta < b$, where k_0 is fixed. Thus $|L(z)| \leq |\ln r| + C$ for some fixed C, so that $L(z) = O(1/z)$ for $z \to 0$ in U: the (multivalued) logarithm function has moderate growth in sectors.

Exercice 8.2.4 Prove that z^α has moderate growth in sectors while $e^{1/z}$ has not.

Basic and obvious properties. For all the following, proof is left as an (easy) exercice:

- Multivalued functions with moderate growth in sectors are closed under linear combinations and multiplication: they form a $\mathbf{C}$-algebra.
- The matricial function $z^A = \exp(A \log z)$, where $A \in \mathrm{Mat}_n(\mathbf{C})$, has moderate growth in sectors, meaning that all its coefficients have: this follows from the example, the exercice, and the previous property.
- For a uniform function on $\dot{D}$, the condition of moderate growth in sectors is equivalent to being meromorphic after the corollary of theorem 3.2.1.

A non obvious property is that multivalued functions with moderate growth in sectors actually form a *differential* $\mathbf{C}$-algebra.

Lemma 8.2.5 *If g is analytic and bounded in $U' := \dot{D} \cap S_{a',b'}$, then, for any $a, b \in \mathbf{R}$ such that $a' < a < b < b'$, the function zg' is bounded in $U := \dot{D} \cap S_{a,b}$.*

Proof. - By elementary geometry, there exists a constant $C > 0$ such that:

$$\forall z \in S_{a,b}\ ,\ d(z, \partial S_{a',b'}) \geq C\,|z|\,.$$

Now, assuming $|g| \leq M$ on U and using *Cauchy estimates* (Ahlfors, chap 4, §2.3 p. 122), one gets:

$$|g'(z)| \leq \frac{M}{d(z, \partial S_{a',b'})} \Longrightarrow |cg'(z)| \leq \frac{M}{C}.$$

□

Theorem 8.2.6 *If f has moderate growth in sectors on $\dot{D}$, so has f'.*

Proof. - For a given sector $S_{a,b}$, fix a slightly bigger one $S_{a',b'}$ with $a' < a < b < b'$ such that $b' - a' < 2\pi$. Choose N such that $g := z^N f$ is bounded on $\dot{D} \cap S_{a',b'}$. Then $zg' - Ng$ is bounded on $\dot{D} \cap S_{a,b}$ after the lemma, so $f' = (zg' - Ng)z^{-N-1} = O(z^{-N-1})$ on $\dot{D} \cap S_{a,b}$. □

Exercice 8.2.7 Use trigonometry to compute the constant C in the lemma.

8.3 Moderate growth condition for solutions of a system

Let $A \in \text{Mat}_n(\mathbf{C}(\{z\}))$ be holomorphic on the punctured disk $\dot{D} := \overset{\circ}{\text{D}}(0,R) \setminus \{0\}$. Let $\mathcal{X}$ a fundamental matricial solution of the system S_A at $a \in \dot{D}$. Then, if $\mathcal{X}$ has moderate growth in sectors, the same is true for all fundamental matricial solutions of S_A at any point of $\dot{D}$ (because thay can be obtained from $\mathcal{X}$ by analytic continuation and multiplication by a constant matrix), and therefore also for all (vector) solutions.

Definition 8.3.1 We say that S_A *has a regular singular point at* 0, or that S_A *is regular singular at* 0, if it has a fundamental matricial solution at some point with moderate growth in sectors.

The following properties are then obvious:

1. The system S_A is regular singular at 0 if, and only if all its (vector) solutions at some point have moderate growth in sectors.
2. If S_A has a regular singular point at 0 and if $A \sim B$ (meromorphic equivalence at 0), then S_B is regular singular at 0. Indeed, if $B = F[A]$ with $F \in \text{GL}_n(\mathbf{C}(\{z\}))$ and if $\mathcal{X}$ is a fundamental matricial solution of S_A, then $F\mathcal{X}$ is fundamental matricial solution of S_B and it has moderate growth in sectors.

Example 8.3.2 If $A = z^{-1}C$ with $C \in \text{Mat}_n(\mathbf{C})$, then z^C is a fundamental matricial solution and it has moderate growth in sectors, so 0 is a regular singular point for S_A.

This example is in some sense “generic”:

Theorem 8.3.3 *If the system $X' = AX$ is regular singular at* 0*, then there is a matrix $C \in \text{Mat}_n(\mathbf{C})$ such that $A \sim z^{-1}C$.*

Proof. - Let $\mathcal{X}$ a fundamental matricial solution at some point $a \in \dot{D}$. The result of analytic continuation along the fundamental loop λ is $\mathcal{X}^\lambda = \mathcal{X}M$ for some invertible monodromy matrix $M \in \text{GL}_n(\mathbf{C})$. From section 4.4, we know that there exists a matrix $C \in \text{Mat}_n(\mathbf{C})$ such that $e^{2i\pi C} = M$. Let $F := \mathcal{X}z^{-C}$. Then $F^\lambda = \mathcal{X}^\lambda (z^{-C})^\lambda = \mathcal{X}Me^{-2i\pi C}z^{-C} = \mathcal{X}MM^{-1}z^{-C} = \mathcal{X}z^{-C} = F$, that is, F is uniform; but also $\mathcal{X}$ and z^{-C} having moderate growth, so has F. Therefore, $F \in \text{GL}_n(\mathbf{C}(\{z\}))$; and of course $F[z^{-1}C] = A$. $\Box$

Definition 8.3.4 We say that S_A *has a singularity of the first kind at* 0, or that 0 *is a singularity of the first kind for S_A*, if A has at most a simple pole at 0, *i.e.* $zA \in \text{Mat}_n(\mathbf{C}\{z\})$. We also sometimes say improperly for short that S_A *is of the first kind.*

Examples 8.3.5

1. The rank one system $zf' = \alpha f$ is of the first kind.
2. If $a_1, \ldots, a_n$ have simple pole at 0 and if $A := A_{\underline{a}}$, then S_A has a singularity of the first kind at 0.
3. Suppose p and q are holomorphic at 0. We vectorialize the equation $f'' + (p/z)f' + (q/z^2)f = 0$ by putting $X := \begin{pmatrix} f \\ zf' \end{pmatrix}$ so that $X' = AX$ with $A = z^{-1}\begin{pmatrix} 0 & 1 \\ -q & 1-p \end{pmatrix}$: this is a system of the first kind.

Theorem 8.3.6 *The system S_A is regular singular at* 0 *if, and only if it is meromorphically equivalent to a system having a singularity of the first kind at* 0.

Proof. - The previous theorem shows a stronger version of one implication. For the converse implication, it will be enough to construct a fundamental matricial solution for a system having a singularity of the first kind at 0, and to prove that this solution has moderate growth in sectors. This will be done in the next section (theorem 8.4.1). □

The main difficulty in using this theorem is that it is not easy to see if a given system is meromorphically equivalent to one of the first kind.

8.4 Resolution and monodromy of regular singular systems

The following result and method of resolution are, in essence, due to Fuchs and Frobenius.

Theorem 8.4.1 *The system $X' = z^{-1}AX$, $A \in Mat_n(\mathbf{C}\{z\})$, has a fundamental matricial solution of the form $\mathcal{X} = Fz^C$, $F \in GL_n(\mathbf{C}(\{z\}))$ and $C \in Mat_n(\mathbf{C})$.*

Proof. - This amounts to say that F is a meromorphic gauge transformation from $z^{-1}C$ to $z^{-1}A$, *i.e.* $zF' = AF - FC$. This "simplification" of A into a constant matrix $C = F^{-1}AF - zF^{-1}F'$ will be achieved in two main steps.

Elimination of resonancies. *Resonancies*[2] are here occurrences of pairs of distinct eigenvalues $\lambda \neq \mu \in \mathrm{Sp}(A(0))$ such that $\mu - \lambda \in \mathbf{N}$. The second step of the resolution of our system (using a Birkhoff gauge transformation) will require that there be no resonancies, so we begin by eliminating them. To do that, we alternate constant gauge transformations (with matrix in $\mathrm{GL}_n(\mathbf{C})$) and "shearing" gauge transformations: these are transformations with diagonal matrices $S_{k,l} := \mathrm{Diag}(z, \ldots, z, 1, \ldots, 1)$ (k times z and l times 1).

We begin by triangularizing $A(0) \in \mathrm{Mat}_n(\mathbf{C})$. Note that if $P \in \mathrm{GL}_n$, then $P' = 0$ and the gauge transformation $P[A] = PAP^{-1}$ is just a conjugation; also, we then have $P[A](0) = PA(0)P^{-1}$, which we can choose to be upper triangular. Assume that there are resonancies and choose $\lambda \neq \mu \in \mathrm{Sp}(A(0))$ such that $m := \mu - \lambda \in \mathbf{N}^*$ is as big as possible. We can choose P such that the eigenvalue μ is in the top left block of $PA(0)P^{-1}$:

$$A' := PAP^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ where } a(0) \in \mathrm{Mat}_k(\mathbf{C}), \mathrm{Sp}(a(0)) = \{\mu\};$$

$$b(0) = 0_{k,l}; \quad c(0) = 0_{l,k}; \quad d(0) \in \mathrm{Mat}_l(\mathbf{C}), \mu \notin \mathrm{Sp}(d(0)).$$

We now apply the shearing gauge transformation $S_{k,l}$ to A' and get:

$$A'' := S_{k,l}[A'] = \begin{pmatrix} a - I_k & z^{-1}b \\ zc & d \end{pmatrix},$$

so that $A''(0)$ has the same eigenvalues as $A(0)$, except that all μ have been transformed to $\mu - 1 = \lambda + m - 1$: the total quantity of resonancies has strictly decreased. Iterating the process, we get rid of all resonancies.

[2]The name is due to a similar situation in the study of periodic differential equations.

Birkhoff gauge transformation. A *Birkhoff gauge transformation* has matrix $F = I_n + zF_1 + \cdots$ (it can be formal or convergent).

Proposition 8.4.2 *If $A \in Mat_n(\mathbf{C}[[z]])$ has no resonancies, then there is a unique formal Birkhoff gauge transformation F such that $F^{-1}AF - zF^{-1}F' = A(0)$.*

Proof. - The condition $F = I_n + zF_1 + \cdots$, *i.e.* $F(0) = I_n$, implies that F is invertible, so that the relation $F^{-1}AF - zF^{-1}F' = A(0)$ is equivalent to $zF' = AF - FA(0)$. Writing $A = A_0 + zA_1 + \cdots$, this equivalent to $F_0 = I_n$ and, for $k \geq 1$:

$$kF_k = A_0F_k + \cdots + A_kF_0 - F_kA_0 \Longleftrightarrow F_k(A_0 + kI_n) - A_0F_k = A_1F_{k-1} + \cdots + A_kF_0.$$

Note that, by the assumption of non resonancy, the matrices $A_0 + kI_n$ and A_0 have no common eigenvalue for $k \geq 1$. Using the following lemma, we conclude that the F_k are unique and can be recursively calulated using the following formula:

$$F_k := \Phi^{-1}_{A_0, A_0+kI_n}(A_1F_{k-1} + \cdots + A_kF_0).$$

□

Lemma 8.4.3 *Let $P \in Mat_n(\mathbf{C})$ and $Q \in Mat_p(\mathbf{C})$ and define the linear map $\Phi_{P,Q}(X) := XQ - PX$ from $Mat_{n,p}(\mathbf{C})$ into itself. Then, the eigenvalues of $\Phi_{P,Q}$ are the $\mu - \lambda$, where $\lambda \in Sp(P)$ and $\mu \in Sp(Q)$. In particular, if P and Q have no common eigenvalue, then $\Phi_{P,Q}$ is bijective.*

Proof. - If P and Q each are in triangular form, then $\Phi_{P,Q}$ is triangular in the canonical basis of $\mathrm{Mat}_{n,p}(\mathbf{C})$ put in the right order. In general, if P and P' are conjugate and Q and Q' are conjugate, then $\Phi_{P,Q}$ and $\Phi_{P',Q'}$ are conjugate. □

Proposition 8.4.4 *In the previous proposition, if A moreover converges: $A \in Mat_n(\mathbf{C}\{z\})$, then the Birkhoff gauge transformation also converges: $F \in GL_n(\mathbf{C}\{z\})$.*

Proof. - We use very basic functional analysis (normed vector spaces). When k grows, $\Phi_{A_0,A_0+kI_n} \sim kId$ in $\mathrm{End}(\mathrm{Mat}_{n,p}(\mathbf{C}))$, so for an adequate norm: $\left|\left|\left|\Phi^{-1}_{A_0,A_0+kI_n}\right|\right|\right| \sim 1/k$. Therefore, there is a constant $D > 0$ such that:

$$|||F_k||| \leq (D/k) \sum_{i=0}^{k-1} |||A_{k-i}||| \, |||F_i||| \, .$$

Also, by hypothesis of convergence, there exists $C, R > 0$ such that $\left|\left|\left|A_j\right|\right|\right| \leq CR^{-j}$. Putting $g_k := R^k\,|||F_k|||$, one sees that $g_k \leq (CD/k) \sum\limits_{i=0}^{k-1} g_i$. Now, increasing C or D, we can clearly assume that $CD \geq 1$ and an easy induction gives $g_k \leq (CD)^k$, whence $|||F_k||| \leq (CD/R)^k$. □

This ends the proof of theorem 8.4.1. □

Corollary 8.4.5 *If we choose as a fundamental matricial solution $\mathcal{X} := Fz^C$ as constructed in the theorem, we find the monodromy matrix:*

$$M_{[\lambda]} = \mathcal{X}^{-1}\mathcal{X}^{\lambda} = z^{-C}F^{-1}Fz^Ce^{2i\pi C} = e^{2i\pi C}.$$

Exercice 8.4.6 Fill in the details in the proof of the lemma.

The local Riemann-Hilbert correspondance. Remember that we saw in definition 7.6.14 when two linear representations are equivalent. In the case of the group $\pi_1(\mathbf{C}^*, a)$, a linear representation is completely characterized by the image $M \in \mathrm{GL}_n(\mathbf{C})$ of the standard generator (the homotopy class of the fundamental loop λ); and the representations characterized by $M, N \in \mathrm{GL}_n(\mathbf{C})$ are equivalent if, and only if, the matrices M and N are conjugate, *i.e.* $N = PMP^{-1}$, $P \in \mathrm{GL}_n(\mathbf{C})$. When we apply these general facts to the monodromy representation of a particular system, we identify $\pi_1(\mathbf{C}^*, a)$ with $\pi_1(\dot{D}, a)$.

Theorem 8.4.7 *Associating to a regular singular system $X' = z^{-1}AX$, $A \in Mat_n(\mathbf{C}\{z\})$ one of its monodromy representations $\rho : \pi_1(\mathbf{C}^*, a) \to GL(\mathcal{F}_a)$ yields a bijection between meromorphic equivalence classes of regular singular systems on the one hand and isomorphism classes of linear representations of the fundamental group on the other hand.*

Proof. - Suppose $A, B \in \mathrm{Mat}_n(\mathbf{C}\{z\})$ define equivalent systems: $z^{-1}A \sim z^{-1}B$, and let F the corresponding gauge transformation. Then, if $\mathcal{X}$, $\mathcal{Y}$ respectively are fundamental matricial solutions for these two systems, one has $F\mathcal{X} = \mathcal{Y}P$ for some $P \in \mathrm{GL}_n(\mathbf{C})$ and their respective monodromy matrices $M := \mathcal{X}^{-1}\mathcal{X}^\lambda$ and $N := \mathcal{Y}^{-1}\mathcal{Y}^\lambda$ satisfy the relation:

$$N = (F\mathcal{X}P^{-1})^{-1}(F\mathcal{X}P^{-1})^\lambda = P\mathcal{X}^{-1}F^{-1}(F\mathcal{X}^\lambda P^{-1}) = PMP^{-1}.$$

Therefore, the mapping from classes of systems to classes of representations mentioned in the theorem is well defined.
Conversely, if we are given A and B, the fundamental matricial solutions $\mathcal{X}$, $\mathcal{Y}$, the monodromy matrices $M := \mathcal{X}^{-1}\mathcal{X}^\lambda$ and $N := \mathcal{Y}^{-1}\mathcal{Y}^\lambda$ and a conjugating matrix $P \in \mathrm{GL}_n(\mathbf{C})$ such that $N = PMP^{-1}$, setting $F := \mathcal{Y}P\mathcal{X}^{-1}$, we see first that:

$$F^\lambda = \mathcal{Y}^\lambda P(\mathcal{X}^\lambda)^{-1} = \mathcal{Y}NPM^{-1}\mathcal{X}^{-1} = \mathcal{Y}P\mathcal{X}^{-1} = F,$$

that is, F is uniform; and since the systems are regular singular, so that $\mathcal{X}$, $\mathcal{Y}$ have moderate growth in sectors, so has F which is therefore a meromorphic gauge transformation relating the two systems. Thus the mapping from classes of systems to classes of representations mentioned in the theorem is injective.
Last, since any $M \in \mathrm{GL}_n(\mathbf{C})$ can be written $e^{2i\pi C}$, we know from the corollary to the previous theorem that the above mapping from classes of systems to classes of representations is surjective.
$\square$

Exercice 8.4.8 In the proof of injectivity above, check rigorously that F has moderate growth in sectors.

8.5 Moderate growth condition for solutions of an equation

We now look for a condition on the functions $a_i(z)$ ensuring that the scalar equation $E_{\underline{a}}$ has a basis of solutions at some point $a \in \dot{D}$ having moderate growth in sectors. Of course, in this case, every basis at every point has moderate growth, and all solutions have moderate growth.

Definition 8.5.1 We say that $E_{\underline{a}}$ *has a regular singular point at* 0, or that $E_{\underline{a}}$ *is regular singular at* 0, if it has a basis of solutions at some point with moderate growth in sectors.

Examples 8.5.2 (i) If all a_i have a simple pole at 0, then the system with matrix $A_{\underline{a}}$ is of the first kind, hence it is regular singular, and $E_{\underline{a}}$ is too.
(ii) If p has a simple pole and q a double pole, then vectorializing with $\begin{pmatrix} f \\ zf' \end{pmatrix}$ yields a system of the first kind, so that $f'' + pf' + qf = 0$ is regular singular at 0.

The last example suggests that we should use the differential operator $\delta := z\dfrac{d}{dz}$ (which is sometimes called "Euler differential operator") instead of the differential operator $D := \dfrac{d}{dz}\cdot$ Both are "derivations", which means that they are $\mathbf{C}$-linear and satisfy "Leibniz rule":

$$
\begin{aligned}
D(fg) &= fD(g) + D(f)g, \\
\delta(fg) &= f\delta(g) + \delta(f)g.
\end{aligned}
$$

Both operate at will on $\mathbf{C}(z)$, on $\mathbf{C}\{z\}$, on $\mathbf{C}(\{z\})$ and even on $\mathbf{C}[[z]]$ and $\mathbf{C}((z))$. We are going to do some elementary differential algebra with them.

The operator $\delta = zD$ is the compositum of operator $D : f \mapsto Df$ and of operator $z : f \mapsto zf$. This composision is not commutative: the operator Dz sends f to $D(zf) = f + zD(f)$, so that one can write: $Dz = zD + 1$, where 1 denotes the identity operator $f \mapsto 1.f = f$.

Exercice 8.5.3 Show that for all $k, l \in \mathbf{N}$, $D^k z^l$ is a linear combination of operators $z^i D^j$.

Lemma 8.5.4 *(i) One has the equality:* $z^k D^k = \delta(\delta - 1)\cdots(\delta - k + 1)$.
(ii) For $k \geq 2$, *the operator* δ^k *is equal to* $z^k D^k +$ *a linear combination of* $zD, \ldots, z^{k-1}D^{k-1}$.

Proof. - (i) can be proved easily by induction, but it is simpler to look at the effect of both sides acting on z^m, $m \in \mathbf{Z}$. Clearly, $\delta(z^m) = mz^m$, so that, for any polynomial $P \in \mathbf{C}[X]$, one has $P(\delta)(z^m) = P(m)z^m$ (these are the classical rules about polyomials in endomorphisms and eigenvectors). On the other hand, we know that $D^k(z^m) = m\cdots(m-k+1)z^{m-k}$, so that $z^k D^k(z^m) = m\cdots(m-k+1)z^m$.
(ii) We now have a triangular system of relations that we can solve recursively:

$$
\begin{cases}
zD = \delta, \\
z^2D^2 = \delta^2 - \delta, \\
z^3D^3 = \delta^3 - 3\delta^2 + 2\delta, \\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{cases}
\Longrightarrow
\begin{cases}
\delta = zD, \\
\delta^2 = z^2D^2 + zD, \\
\delta^3 = z^3D^3 + 3z^2D^2 + zD, \\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{cases}
$$

□

Using these relations, we can transform a differential equation $f^{(n)} + b_1 f^{(n-1)} + \cdots b_n f = 0$ into one involving δ in the following way: rewrite the equation in the more symbolic form $(D^n + b_1 D^{n-1} + \cdots + b_n)f = 0$; then multiply at left by z^n and replace $z^n D^i$ by $z^{n-i}(z^i D^i) = z^{n-i}\delta\cdots(\delta - i + 1)$. The process can be reversed.

Proposition 8.5.5 *Assume $z^n(D^n + b_1 D^{n-1} + \cdots + b_n) = \delta^n + a_1\delta^{n-1} + \cdots + a_n$. Then:*

$$\big(v_0(b_1) \geq -1, \ldots, v_0(b_n) \geq -n\big) \Longleftrightarrow a_1, \ldots, a_n \in \mathbf{C}\{z\}.$$

Proof. - Call T the triangular matrix such that $(1, zD, \ldots, z^n D^n) = (1, \delta, \ldots, \delta^n)T$. According to the lemma above, its coefficients are in $\mathbf{C}$ and its diagonal coefficients are all equal to 1. Then, setting $a_0, b_0 := 1$:

$$\begin{pmatrix} a_n \\ \vdots \\ a_0 \end{pmatrix} = T \begin{pmatrix} b_n z^n \\ \vdots \\ b_0 \end{pmatrix}.$$

which shows that the a_i are linear combinations with constant coefficients of the $b_i z^i$, and conversely. □

The following criterion shows that, contrary to the case of systems, it is very easy to check if all the solutions of an equation have moderate growth at 0. This is a justification for the utility of cyclic vectors.

Theorem 8.5.6 (Fuchs criterion) *The equivalent equations $f^{(n)} + b_1 f^{(n-1)} + \cdots b_n f = 0$ and $\delta^n f + a_1\delta^{n-1} f + \cdots + a_n f = 0$ are regular singular at 0 if, and only if, $v_0(b_i) \geq -i$ for $i = 1, \ldots, n$; or equivalently: $a_1, \ldots, a_n \in \mathbf{C}\{z\}$.*

Proof. - It follows from the proposition above that the two criteria stated are indeed equivalent. Suppose they are verified. Then the system obtained by the vectorialisation $X := \begin{pmatrix} f \\ \delta f \\ \vdots \\ \delta^{n-1} f \end{pmatrix}$ has the matrix $z^{-1}A_{\underline{a}}$, so it is of the first kind and thus regular singular; so the equation is also regular singular.

Conversely assume that the equation is regular singular. If $n = 1$, the equation can be seen as a system $f' = -b_1 f$ of rank 1 and we apply the criterion for systems: there exists a meromorphic non zero u such that $u[-b_1] = -b_1 + u'/u$ has a simple pole at 0. But u'/u itself has a simple pole at 0, so b_1 also has, which is the desired conclusion. The proof for the general cas $n \geq 2$ will be by induction and it will require several steps.

Step 1. So we suppose that $n \geq 2$, that $a_1, \ldots, a_n \in \mathbf{C}(\{z\})$ and that all the solutions of the equation $\delta^n f + a_1\delta^{n-1} f + \cdots + a_n f = 0$ have moderate growth.

Lemma 8.5.7 *The equation has at least one solution of the form $f = uz^\alpha$, where $\alpha \in \mathbf{C}$ and $u = 1 + u_1 z + \cdots \in \mathbf{C}\{z\}$.*

Proof. - Let $(f_1, \ldots, f_n)$ be a fundamental system of solutions at some point and $M \in \mathrm{GL}_n(\mathbf{C})$ the matrix of its monodromy along the fundamental loop λ, *i.e.* $(f_1^\lambda, \ldots, f_n^\lambda) = (f_1, \ldots, f_n)M$. We triangularize M: there is $P \in \mathrm{GL}_n(\mathbf{C})$ such that $M = PTP^{-1}$ and T is upper triangular; call β its first diagonal coefficient. Since M is invertible, so is T and $\beta \neq 0$, so that we can write $\beta = e^{2i\pi\alpha}$ for some $\alpha \in \mathbf{C}$. Then $(g_1, \ldots, g_n) := (f_1, \ldots, f_n)P$ is a fundamental system of solutions and

$$(g_1^\lambda, \ldots, g_n^\lambda) = (f_1^\lambda, \ldots, f_n^\lambda)P = (f_1, \ldots, f_n)MP = (g_1, \ldots, g_n)P^{-1}MP = (g_1, \ldots, g_n)T,$$

which implies in particular that $g_1^\lambda = \beta g_1$. Thus, $g_1 z^{-\alpha}$ is uniform and has moderate growth (by the assumption of regularity), so it is a meromorphic function $cz^m u$ with u as indicated in the statement of the lemma. Then, changing α to $\alpha + m$, we find that g_1 is a solution af the form required. □

Step 2. We this have a particular solution of the form $f_0 = uz^\alpha$ as above and we look for the equation the solutions of which are the f/z^α, where f is a solution of $\delta^n f + a_1\delta^{n-1} f + \cdots + a_n f = 0$. This a change of unknown function. So we put $f = z^\alpha g$, and, noticing that $\delta.z^\alpha = z^\alpha.(\delta+\alpha)$, we obtain:

$$\delta^n(z^\alpha g) + a_1\delta^{n-1}(z^\alpha g) + \cdots + a_n(z^\alpha g) = z^\alpha\big((\delta+\alpha)^n g + a_1(\delta+\alpha)^{n-1} g + \cdots + a_n g\big),$$

whence a new equation $\delta^n g + b_1\delta^{n-1} g + \cdots + b_n g = 0$, where the a_i, b_i are related by the formula:

$$X^n + b_1 X^{n-1} + \cdots + b_n = (X+\alpha)^n + a_1(X+\alpha)^{n-1} + \cdots + a_n.$$

In particular, the a_i are holomorphic if, and only if, the b_i are. And of course, since $f = z^\alpha g$ has moderate growth if, and only if, g has moderate growth, we see that the equation $\delta^n f + a_1\delta^{n-1} f + \cdots + a_n f = 0$ is regular singular if, and only if, the equation $\delta^n g + b_1\delta^{n-1} g + \cdots + b_n g = 0$ is. Therefore, we are led to prove the theorem for the latter equation. But we know that this one has a particular solution $u = 1 + u_1 z + \cdots \in \mathbf{C}\{z\}$.

Exercice 8.5.8 Is $f = z^\alpha g$ a gauge transformation in the sense of section 7.6 ?

Step 3. We are now going to operate a euclidean division of polynomials, but in a non commutative setting !

Lemma 8.5.9 *Let $v \in \mathbf{C}(\{z\})$. Then every differential operator $\delta^m + p_1\delta^{m-1} + \cdots + p_m$ with $p_1, \ldots, p_m \in \mathbf{C}(\{z\})$ can be written in the form $(\delta^{m-1} + q_1\delta^{m-2} + \cdots + q_{m-1})(\delta - v) + w$, where $w, q_1, \ldots, q_{m-1} \in \mathbf{C}(\{z\})$.*

Proof. - We note first that $\delta^i - \delta^{i-1}(\delta - v)$ is a sum of terms $r_j\delta^j$, $j = 0, \ldots, i-1$, with all $r_j \in \mathbf{C}(\{z\})$. From this we deduce by induction that the theorem is true for each differential operator δ^i. Then we get the conclusion by linear combination. □

Now we apply the lemma with $v := \delta(u)/u$ and get:

$$\delta^n + b_1\delta^{n-1} + \cdots + b_n = (\delta^{n-1} + c_1\delta^{n-2} + \cdots + c_{n-1})(\delta - v) + w.$$

Applying this to u, and noting that $(\delta - v)u = \delta(u) - vu = 0$, we get $wu = 0$ so that $w = 0$. Therefore, we have the equality:

$$\delta^n + b_1\delta^{n-1} + \cdots + b_n = (\delta^{n-1} + c_1\delta^{n-2} + \cdots + c_{n-1})(\delta - v).$$

Step 4. Now we will apply the induction hypothesis to the new operator. We note that, if $g_1, \ldots, g_n$ are a basis of solutions of the equation $\delta^n g + b_1\delta^{n-1}g + \cdots + b_n g = 0$, then the $h_i := (\delta - v)g_i$ are solutions of $\delta^{n-1}h + c_1\delta^{n-2}h + \cdots + c_{n-1}h = 0$, and, of course, they have moderate growth (since the g_i and v have). We choose the basis such that $g_n = u$. Then $h_1, \ldots, h_{n-1}$ are linearly independant. Indeed, if $\lambda_1 h_1 + \cdots + \lambda_{n-1}h_{n-1} = 0$, then the function $g := \lambda_1 g_1 + \cdots + \lambda_{n-1}g_{n-1}$ satisfies $\delta(g) = vg$; but since $v = \delta(u)/u$, this implies $\delta(g/u) = 0$, whence $\lambda_1 g_1 + \cdots + \lambda_{n-1}g_{n-1} = \lambda g_n$, which is only possible if all $\lambda_i = 0$. Therefore, $h_1, \ldots, h_{n-1}$ are a basis of solutions of $\delta^{n-1}h + c_1\delta^{n-2}h + \cdots + c_{n-1}h = 0$ and, since they have moderate growth, this is a regular singular equation. By the inductive hypothesis of the theorem, all $c_i \in \mathbf{C}\{z\}$. But then, from $\delta^n + b_1\delta^{n-1} + \cdots + b_n = (\delta^{n-1} + c_1\delta^{n-2} + \cdots + c_{n-1})(\delta - v)$ and the fact that $v \in \mathbf{C}\{z\}$, we conclude that all $b_i \in \mathbf{C}\{z\}$, which ends the induction step. $\Box$

8.6 Resolution and monodromy of regular singular equations

We shall only present the basic cases and examples. The general case is rather complicated because of resonancies, which anyway are exceptional. For the complete algorithms (mostly due to Fuchs and Frobenius), see the references given at the beginning of the chapter.

We start with equation $\delta^n f + a_1\delta^{n-1}f + \cdots + a_n f = 0$, which we assume to be regular singular at 0, *i.e.* $a_1, \ldots, a_n \in \mathbf{C}\{z\}$. As we saw in the proof of theorem 8.5.6, there certainly is a solution of the form $f = z^\alpha u$, $\alpha \in \mathbf{C}$, $u = 1 + u_1 z + \cdots \in \mathbf{C}\{z\}$. Actually, it even follows from the argument given there that, if the monodromy is semi-simple, there is a whole basis of solutions of this form. From the relation we had proved: $\delta z^\alpha = z^\alpha(\delta + \alpha)$, we draw the equation:

$$(\delta+\alpha)^n u + a_1(\delta+\alpha)^{n-1}u + \cdots + a_n u = \delta^n u + b_1\delta^{n-1}u + \cdots + b_n u = 0,$$

where, setting $P(X) := X^n + a_1X^{n-1} + \cdots + a_n$ and $Q(X) := X^n + b_1X^{n-1} + \cdots + b_n$, we have $Q(X) = P(X+\alpha)$. Now In the equation $Q(\delta)(u) = 0$, all terms are multiple of z and thus vanish at 0, except maybe $b_n u$, so that: $b_n(0)u(0) = 0$. But $u(0) = 1$, so that $b_n(0) = 0$, whence the *indicial equation*:

$$\alpha^n + a_1(0)\alpha^{n-1} + \cdots + a_n(0) = 0.$$

This is a *necessary* condition for α to be a possible exponent. It can be proved that, if α is a non resonant root of this equation, that is no $\alpha + k$, $k \in \mathbf{N}^*$, is a root, then the condition is also sufficient. We only prove a slightly weaker result.

Theorem 8.6.1 *If the indicial equation has n distinct and non resonant roots $\alpha_1, \ldots, \alpha_n$ (so that $\alpha_i - \alpha_j \notin \mathbf{Z}$ for $i \neq j$), then there is a fundamental system of solutions of the form $f_i = z_i^\alpha u_i$, $u_i \in \mathbf{C}\{z\}$, $u_i(0) = 1$.*

Proof. - The matrix of the corresponding system is $A := A_{\underline{a}}$ and the characteristic polynomial of the companion matrix $A(0)$ is the one defining the indicial equation: therefore, by assumption, $A(0)$ is non resonant in the sense of section 8.4. So the system has a fundamental matricial solution $Fz^{A(0)}$, where F is a Birkhoff matrix. Since $A(0)$ is semi-simple, we can write $A(0) = P\mathrm{Diag}(\alpha_1, \ldots, \alpha_n)P^{-1}$ and conclude that $FP\mathrm{Diag}(z^{\alpha_1}, \ldots, z^{\alpha_n})$ is another fundamental matricial solution. Its first line gives the fundamental system of solutions of the required form. $\Box$

From there on, we only study examples !

Example 8.6.2 We consider $\delta^2 f - \delta f = 0$. The indicial equation is $\alpha^2 - \alpha = 0$, whence two roots 0 and 1. In this case both roots yield solutions, because 1 and z are indeed solutions (and a fundamental system of solutions since their wronskian is 1).
Actually, the system obtained by vectorialisation has constant matrix $A := \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $A^2 = A$, so that we easily compute $z^A = \begin{pmatrix} 1 & z-1 \\ 0 & z \end{pmatrix}$, whence the fundamental system of solutions $(1, z-1)$.

Example 8.6.3 We consider $\delta^2 f - \delta f - zf = 0$. The indicial equation is again $\alpha^2 - \alpha = 0$, whence two roots 0 and 1. In this case there is a problem. Indeed, looking for a power series solution $\sum\limits_{n\geq 0} f_n z^n$, we find the equivalent relations $(n^2 - n) f_n - f_{n-1} = 0$. This implies that $f_0 = 0$, f_1 is free and can be taken equal to 1, and the other coefficients are recursively computed: $f_n = \dfrac{1}{n!(n-1)!}$ for $n \geq 1$. This has the required form for $\alpha = 1$, a non resonant root, but the resonant root 0 gave nothing.
To see why, we vectorialize the equation: here, $A := \begin{pmatrix} 0 & 1 \\ -z & 1 \end{pmatrix}$. But $A(0)$ has roots 0 and 1, it is resonant and we must transform it before solving (since it is not constant, computing z^A would not make any sense). The shearing transform $\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$ gives $B := F[A] = \begin{pmatrix} 0 & z \\ -1 & 0 \end{pmatrix}$. But now, $B(0)$ is non resonant but nilpotent and $z^{B(0)}$ contains a logarithm, so the fundamental matricial solutions of B and A also do.

Example 8.6.4 Consider the divergent series $f := \sum\limits_{n\geq 1} (n-1)! z^n$ (Euler series[3]). It satisfies the non homogeneous first order equation $f = z + z\delta f$, that is, $(1 - z\delta) f = z$. Since $z = \delta z$, the series f is solution of $(1-\delta)(1-z\delta) f = 0$, that is $z\delta^2 f - \delta f + f = 0$, an *irregular* equation.

Example 8.6.5 Bessel equation is $z^2 f'' + z f' + (z^2 - \alpha^2) f$. It is regular (*i.e.* it has only ordinary points) in $\mathbf{C}^*$ an it has a regular singular point at 0. With the Euler differential operator, the equation becomes $\delta^2 f + (z^2 - \alpha^2) f = 0$. The indicial equation is (using by necessity the letter x for the unknown) $x^2 - \alpha^2$, which has roots $\pm\alpha$. We assume that $2\alpha \notin \mathbf{Z}$, so that both exponents give rise to a solution. For instance, putting $f = z^\alpha g$ gives rise to the equation $(\delta + \alpha)^2 g + (z^2 - \alpha^2) g = 0$, *i.e.* $\delta^2 g + 2\alpha\delta g + z^2 g = 0$. We look for g in the form $g = g_0 + g_1 z + \cdots$, with $g_0 = 1$. This gives for all $n \geq 0$ the relation: $(n^2 + 2\alpha n) g_n + g_{n-2} = 0$ (with the natural convention that $g_{-1} = g_{-2} = 0$) so that g_1, and then all g_n with odd index n are 0. For even indexes, setting $h_n := g_{2n}$ we find $h_0 = 1$ and $h_n = \dfrac{-1}{4n(n+\alpha)} h_{n-1}$ for $n \geq 1$, whence $h_n = \dfrac{(-1/4)^n}{n!(\alpha+1)\cdots\alpha+n)}\cdot$ In the end, we get the solution:

$$f(z) = z^\alpha \sum_{n\geq 0} \frac{(-1/4)^n}{n!(\alpha+1)\cdots(\alpha+n)} z^{2n}.$$

[3]For a very interesting account of the importance of this series, see the monograph "Séries divergentes et théories asymptotiques" of Jean-Pierre Ramis.

It is customary to consider a constant multiple of this solution, the *Bessel function*:

$$J_\alpha(z) := (z/2)^\alpha \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} (z/2)^{2n}.$$

Here, the Gamma function[4] Γ is an analytic function on $\mathbf{C} \setminus (-\mathbf{N})$, which satisfies the functional equation $\Gamma(z+1) = z\Gamma(z)$, from which one draws immediately that $\Gamma(n+\alpha+1) = (\alpha+1)\cdots(\alpha+n)\Gamma(\alpha+1)$ and then that $f(z) = 2^\alpha\Gamma(\alpha+1)J_\alpha(z)$.

Exercice 8.6.6 What can be said of Bessel equation when $2\alpha \in \mathbf{Z}$? In all cases, how does it behave at infinity ?

Example 8.6.7 The Airy function is defined, for real x, by $Ai(x) := \dfrac{1}{\pi} \int\limits_0^{+\infty} \cos(t^3/3 + xt)\,dt$. It is not difficult to prove that it is well defined and satisfies the differential equation $Ai''(x) = xAi(x)$. So we decide to study the complex differential equation $f'' = xf$. It is regular on $\mathbf{C}$, from which one deduces that it has a fundamental system of uniform solutions in $\mathbf{C}$, so that The Airy function can be extended to an entire function. To study it at infinity, we set $w := 1/z$ and $g(w) := f(z) = f(1/w)$, so that $g'(w) = -w^{-2}f'(1/w)$ and $g''(w) = w^{-4}f''(1/w) + 2w^{-3}f'(1/w)$. We end with the differential equation: $g'' + 2w^{-1}g' - w^{-5}g = 0$, which is irregular at $w = 0$. Actually, the asymptotic behaviour of the Airy function at infinity was the origin of the discovery by Stokes of the so-called “Stokes phenomenon” (see the book of Ramis already quoted).

Example 8.6.8 Put $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) = \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ (using the Gamma function introduced above). For instance, $(1)_n = n!$. The *hypergeometric series of Euler-Gauss* is defined as:

$$F(\alpha,\beta,\gamma;z) := \sum_{n\geq 0} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} z^n.$$

We consider α,β,γ as parameters and z as the variable, so we will here write for short $F(z)$ instead of $F(\alpha,\beta,\gamma;z)$. We must assume that $\gamma \notin -\mathbf{N}$ for this series to be defined. We also assume that $\alpha,\beta \notin -\mathbf{N}$, so that it is not a polynomial. Then the radius of convergence is 1. The coefficients $f_n := \dfrac{(\alpha)_n(\beta)_n}{n!(\gamma)_n}$ satisfy the recursive relation $(n+1)(n+\gamma)f_{n+1} = (n+\alpha)(n+\beta)f_n$. We multiply both sides by z^{n+1} and sum for all $n \geq 0$. Recalling that $\delta(\sum f_n z^n) = \sum n f_n z^n$ and $\delta^2(\sum f_n z^n) = \sum n^2 f_n z^n$, we obtain the equality:

$$\delta(\delta+\gamma-1)F = z(\delta+\alpha)(\delta+\beta)F,$$

from which follows the differential equation:

$$(1-z)\delta^2 F + (\gamma - 1 - (\alpha+\beta)z)\delta F - \alpha\beta z F = 0.$$

This equation is regular singular at 0. The indicial equation is $x^2 + (\gamma-1)x = 0$. If we assume that $\gamma \notin \mathbf{Z}$, we find that there is as fundamental basis made up of a power series with constant term 1 (this is $F(z)$) and of a solution $z^{1-\gamma}G(z)$, where G is a power series with constant term 1. The study of the hypergeometric equation will be continued in the next chapter.

Exercice 8.6.9 Show that G is itself a hypergeometric series with different parameters.

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[4]More will be said on the Gamma function in the next chapter.

Chapter 9

Hypergeometric series and equations

For all complements about this chapter, the following books are warmly recommended:

- Anosov and Bolibrukh, "The Riemann-Hilbert problem";
- Iwasaki, Kimura, Shimomura and Yoshida, "From Gauss to Painlevé" (abreviated GP in the text);
- Whittaker and Watson, "A course of modern analysis" (abreviated WW in the text);
- Yoshida, "Fuchsian differential equations".

9.1 Fuchsian equations and systems

Fuchsian systems.

Definition 9.1.1 A *fuchsian system* is a meromorphic differential system on $\mathbf{S}$ which has only singularities of the first kind.

First we look at the meromorphy condition. The system $X' = AX$ is meromorphic on $\mathbf{C}$ if $A \in \mathrm{Mat}_n(\mathcal{M}(\mathbf{C}))$. At infinity, we put $w := 1/z$ and $Y(w) := X(z)$ so that $Y'(w) = B(w)Y(w)$, where $B(w) = -w^{-2}A(w^{-1})$. Thus, $X' = AX$ is meromorphic at infinity if $-w^{-2}A(w^{-1})$ is meromorphic at $w = 0$, that is if A is meromorphic at infinity. The condition is therefore $A \in \mathrm{Mat}_n(\mathcal{M}(\mathbf{S}))$, *i.e.* $A \in \mathrm{Mat}_n(\mathbf{C}(z))$ according to theorem 7.1.7. Next we look at the condition on singularities. The rational matrix A has a finite number of poles on $\mathbf{C}$, and they must be simple poles. Call them $a_1, \ldots, a_m$ (of course it is possible that $m = 0$). Then by standard properties of rational functions, one can write $A = \sum\limits_{k=1}^{m} \dfrac{1}{z - a_k} A_k + C$, where C has polynomial coefficients. Then, we want $B(w) = -w^{-2}A(w^{-1})$ to have a simple pole at $w = 0$. But $B(w) = \sum\limits_{k=1}^{m} \dfrac{-1}{w(1 - a_k w)} A_k - w^{-2}C(w^{-1})$, and so we want $w^{-2}C(w^{-1})$ to have a simple pole at $w = 0$, which is only possible if $C = 0$. We have proved:

Proposition 9.1.2 *Fuchsian systems have the form* $X' = \left(\sum\limits_{k=1}^{m} \dfrac{A_k}{z - a_k}\right) X$, *where the* $A_k \in \mathrm{Mat}_n(\mathbf{C})$.

□

The monodromy group of a fuchsian system. Using the local coordinate $v := z - a_k$, the system above has the form $v dX/dv = (A_k +$ multiples of $v)X$. If we suppose that A_k is non resonant, there is a fundamental matricial solution $\mathcal{X}_k = F_k(z-a_k)^{A_k}$, where F_k is a Birkhoff matrix. The local monodromy (*i.e.* calculated along a small positive fundamental loop around a_k) relative to this basis therefore has matrix $e^{2i\pi A_k}$. In the same way, at infinity, $wB(w) = -\sum_{k=1}^{m} A_k +$ multiples of w, so, if $\sum_{k=1}^{m} A_k$ is non resonant, the local monodromy relative to that basis has matrix $e^{-2i\pi \sum_{k=1}^{m} A_k}$.

However, it is difficult to describe the global monodromy from these local data, because all these matrices are relative to different bases. If one fixes a basis, then one will have to use conjugates of the monodromy matrices computed above; and the conjugating matrices are not easy to find.

Example 9.1.3 If $m = 2$, assuming nonresonancy, we have matrices A_1, A_2 and $A_\infty = -A_1 - A_2$. If one fixes a base point a, three small loops $\lambda_1, \lambda_2, \lambda_\infty$ based at a and each turning once around one of the singular points, then there is a relation, for instance $\lambda_\infty \sim \lambda_1\lambda_2$. If one moreover fixes a basis $\mathcal{B}$ of solutions at a, then there are monodromy matrices M_1, M_2, M_∞ such that $\mathcal{B}^{\lambda_1} = \mathcal{B}M_1$, $\mathcal{B}^{\lambda_2} = \mathcal{B}M_2$ and $\mathcal{B}^{\lambda_\infty} = \mathcal{B}M_\infty$. Then, on the one hand, $M_\infty = M_2 M_1$. On the other hand, M_1 is conjugate to $e^{2i\pi A_1}$, M_2 is conjugate to $e^{2i\pi A_2}$, and M_∞ is conjugate to $e^{2i\pi A_\infty}$: these are not equalities and the conjugating matrices are different.

Exception: the abelian cases. If the monodromy group is abelian, the problem of conjugacy disappears. There are a few cases where abelianity is guaranteed. First, if $m = 0$, the system is $X' = 0$ which has trivial monodromy. If $m = 1$, since the fundamental group of $\mathbf{S} \setminus \{a_1, \infty\} = \mathbf{C} \setminus \{a_1\}$ is isomorphic to $\mathbf{Z}$, the monodromy group is generated by $e^{2i\pi A_1}$ (local monodromy at a_1) and $e^{-2i\pi A_1}$. In the same way, if $m = 2$ but there is no singularity at infinity, then $A_1 + A_2 = 0$, the local monodromies at a_1 and a_2 are respectively generated by $e^{2i\pi A_1}$ and $e^{2i\pi A_2} = e^{-2i\pi A_1}$ and the global monodromy is generated by either matrix.

Exercice 9.1.4 Explain topologically why we obtain inverse monodromy matrices in the last two cases .

If $m = 2$ and there is a singularity at infinity, or if $m \geq 3$, the fundamental group of $\mathbf{S} \setminus \{a_1, \ldots, a_m, \infty\}$ is far from being commutative (it is the so-called "free group on m generators"). But, if $n = 1$, the linear group $\mathrm{GL}_1(\mathbf{C}) = \mathbf{C}^*$ is commutative, so the monodromy group is also commutative. This is the case of a scalar equation $f' = \left(\sum_{k=1}^{m} \dfrac{\alpha_k}{z - a_k}\right) f$. Then, without any resonancy condition, the global monodromy group is generated by the $e^{2i\pi\alpha_k}$, $k = 1, \ldots, m$ (thus a subgroup of $\mathbf{C}^*$).

In summary, the first non trivial (non abelian) case will be for $m = 2$, with a singularity at infinity, and $n = 2$. We shall describe it at the end of this section.

Fuchsian equations. We shall rather study the monodromy representation for scalar equations of order $n = 2$.

Definition 9.1.5 A scalar differential equation is said to be *fuchsian* if it is meromorphic on $\mathbf{S}$ and has only regular singularities.

A similar argument to that given for systems shows that a meromorphic equation on $\mathbf{S}$ must have rational coefficients. However, the condition to be fuchsian is a bit more complicated. We shall just need the case $n = 2$ (see the exercice); the general case is desribed in the references given at the beginning of chapter 8.

Exercice 9.1.6 Show that Fuchsian equations of the second order have the form $f'' + pf' + qf = 0$ where $p = \sum_{k=1}^{m} \dfrac{p_k}{z - a_k}$, the $p_k \in \mathbf{C}$, and where $q = \sum_{k=1}^{m} \left(\dfrac{q_k}{(z-a_k)^2} + \dfrac{r_k}{z - a_k} \right)$, the $q_k, r_k \in \mathbf{C}$ and $\sum_{k=1}^{m} r_k = 0$.

It can be proved moreover that, when $n = 2$, $m = 2$ and there is a singularity at infinity, then all fuchsian equations are reducible to the “hypergeometric equations” that we are going to study in this chapter: see the book GP.

Exercice 9.1.7 Show that if the equation has exactly three singularities on $\mathbf{S}$, then using changes of variable $1/z$ and $z + c$, one may always assume that the singularities are at $0, 1, \infty$.

The first non abelian case In summary, the first non trivial (non abelian) case will be for $m = 2$, with a singularity at infinity, and $n = 2$. As we said, we can take the two points a_1, a_2 to be $0, 1$. Therefore, the monodromy representation will be an anti-morphism of groups from $\pi_1(\mathbf{S} \setminus \{0, 1, \infty\}, a)$ to $\mathrm{GL}_2(\mathbf{C})$.

The fundamental group is here a free group on two generators. Once chosen a base point $a \in \mathbf{S} \setminus \{0, 1, \infty\}$, a small loop λ_0 turning positively once around 0 and a small loop λ_1 turning positively once around 1 (both based at a), the group is freely generated by the homotopy classes of these loops. We shall choose $a := 1/2$ and the loops $\lambda_0(t) := \dfrac{1}{2} e^{2i\pi t}$, $\lambda_1(t) := 1 - \dfrac{1}{2} e^{2i\pi t}$, $t \in [0, 1]$.

A representation of $\pi_1(\mathbf{S} \setminus \{0, 1, \infty\}, 1/2)$ in $\mathrm{GL}_2(\mathbf{C})$ is therefore entirely characterized by the matrices $M_0, M_1 \in \mathrm{GL}_2(\mathbf{C})$ which are the respective images of the homotopy classes $[\lambda_0]$, $[\lambda_1]$. Moreover, the generators $[\lambda_0]$, $[\lambda_1]$ being free, the matrices M_0, M_1 may be chosen at will.

The description of the monodromy representation requires the choice of a basis $\mathcal{B}$ of solutions at $1/2$. Then, the effect of monodromy is encoded in matrices $M_0, M_1, M_\infty \in \mathrm{GL}_2(\mathbf{C})$ such that $\mathcal{B}^{\lambda_0} = \mathcal{B} M_0$, $\mathcal{B}^{\lambda_1} = \mathcal{B} M_1$ and $\mathcal{B}^{\lambda_\infty} = \mathcal{B} M_\infty$, where some loop λ_∞ has been chosen. However, there must be a relation btetween the homotopy classes $[\lambda_0]$, $[\lambda_1]$, $[\lambda_\infty]$ and therefore a corresponding relation between the monodromy matrices M_0, M_1, M_∞. We shall take $\lambda_\infty := (\lambda_0.\lambda_1)^{-1}$, whence the relation: $M_\infty M_1 M_0 = I_2$ (remember we have an anti-morphism).

9.2 The hypergeometric series

Definition 9.2.1 The *Pochhammer symbols* are defined, for $\alpha \in \mathbf{C}$ and $n \in \mathbf{N}$, by the formula:

$$(\alpha)_0 := 1 \text{ and, if } n \geq 1\text{ , } (\alpha)_n := \alpha(\alpha + 1) \cdots (\alpha + n - 1).$$

The *hypergeometric series* of Euler-Gauss with parameters $\alpha, \beta, \gamma \in \mathbf{C}$ is the power series:

$$F(\alpha, \beta, \gamma; z) := \sum_{n \geq 0} \frac{(\alpha)_n (\beta)_n}{n! (\gamma)_n} z^n = \sum_{n \geq 0} \frac{(\alpha)_n (\beta)_n}{(1)_n (\gamma)_n} z^n.$$

We must of course require that $\gamma \notin -\mathbf{N}$ so that the denominators do not vanish. We shall also require that $\alpha, \beta \notin -\mathbf{N}$ so that the series is not a polynomial. Last, for reasons that will appear in the next section (to avoid resonancies), we shall require that $\gamma, \alpha - \beta, \gamma - \alpha - \beta \notin \mathbf{Z}$. (The study is possible in these degenerate cases, see the book GP or the book WW).

The coefficients $f_n := \dfrac{(\alpha)_n (\beta)_n}{n! (\gamma)_n}$ satisfy the relation $f_{n+1}/f_n = (n+\alpha)(n+\beta)/(n+1)(n+\gamma)$. Since the right hand side of this equality tends to 1 when $n \to +\infty$, the radius of convergence of the hypergeometric series is 1.

Example 9.2.2 From the obvious formula $(\alpha)_n = (-1)^n n! \binom{-\alpha}{n}$, we draw that $F(\alpha, \beta, \beta; z) = (1-z)^{-\alpha}$ (the generalized binomial series).

Exercice 9.2.3 Show that $\log \frac{1+z}{1-z} = 2zF(1/2, 1, 3/2; z^2)$ and that $\arcsin z = zF(1/2, 1/2, 3/2; z^2)$.

About the Gamma function. The hypergeometric series is related in many ways to the Gamma function of Euler. (On the Gamma function, see the book WW, or most books of complex analysis.) For $\mathrm{Re}\, z > 0$, one can show that the integral:

$$\Gamma(z) := \int_0^{+\infty} e^{-t} t^{z-1}\, dt$$

is well defined and that the function Γ is analytic on the right half plane. Moreover, integration by parts gives the functional equation $\Gamma(z+1) = z\Gamma(z)$. This allows one to extend Γ to the whole complex plane by putting: $\Gamma(z) := \dfrac{1}{(z)_n} \Gamma(z+n)$, where $n \in \mathbf{N}$ is chosen big enough to have $\mathrm{Re}(z+n) > 0$. The extended function is holomorphic on $\mathbf{C} \setminus (-\mathbf{N})$ and has simple poles on $-\mathbf{N}$. It still satisfies the functional equation $\Gamma(z+1) = z\Gamma(z)$, whencee $\Gamma(z+n) = (z)_n \Gamma(z)$.

The hypergeometric series can be written:

$$F(\alpha, \beta, \gamma; z) := \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n \geq 0} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{\Gamma(n+1)\Gamma(n+\gamma)} z^n.$$

Here are some special values related to the Gamma function:

$$\begin{aligned}
\Gamma(1) &= \int_0^{+\infty} e^{-t}\, dt = 1, \\
\Gamma(n) &= (n-1)! \text{ for } n \in \mathbf{N}^*, \\
\Gamma(1/2) &= \int_{-\infty}^{+\infty} e^{-t^2}\, dt = \sqrt{\pi}, \\
\Gamma'(1) &= -\gamma,
\end{aligned}$$

where $\gamma := \lim_{n \to +\infty} (1 + 1/2 + \cdots + 1/n - \ln n)$ is "Euler-Mascheroni constant", a very mysterious number. We shall use other formulas related to the Gamma function in section 9.5.

9.3 The hypergeometric equation

We saw in the previous section that the coefficients $f_n = \dfrac{(\alpha)_n(\beta)_n}{n!(\gamma)_n}$ of $F(\alpha,\beta,\gamma;z)$ satisfy the relation $f_{n+1}/f_n = (n+\alpha)(n+\beta)/(n+1)(n+\gamma)$. From this, recalling that $\delta(\sum f_n z^n) = \sum n f_n z^n$ and that $\delta^2(\sum f_n z^n) = \sum n^2 f_n z^n$, we start with $(n+1)(n+\gamma)f_{n+1} = (n+\alpha)(n+\beta)f_n$ and calculate:

$$\sum_{n\geq 0}(n+1)(n+\gamma)f_{n+1}z^{n+1} = \sum_{n\geq 0}(n+\alpha)(n+\beta)f_n z^{n+1}$$

$$\Longrightarrow \sum_{n\geq 0}(n+1)^2 f_{n+1}z^{n+1} + \sum_{n\geq 0}(\gamma-1)(n+1)f_{n+1}z^{n+1} = z\left(\sum_{n\geq 0} n^2 f_n z^n + \sum_{n\geq 0}(\alpha+\beta)n f_n z^n + \sum_{n\geq 0}\alpha\beta f_n z^n\right)$$

$$\Longrightarrow (\delta^2 + (\gamma-1)\delta)F(\alpha,\beta,\gamma;z) = z(\delta^2 + (\alpha+\beta)\delta + \alpha\beta)F(\alpha,\beta,\gamma;z),$$

that is, the hypergeometric series $F(\alpha,\beta,\gamma;z)$ is solution of the *hypergeometric differential equation with parameters* α,β,γ:

$$HG_{\alpha,\beta,\gamma}: \quad (1-z)\delta^2 F + ((\gamma-1)-(\alpha+\beta)z)\delta F - \alpha\beta z F = 0.$$

We shall also need the form using the standard differential operator $D := d/dz$ instead of δ. Replacing δ by zD and δ^2 by $z^2D^2 + zD$ and then dividing by z, we get the other form of the equation:

$$HG'_{\alpha,\beta,\gamma}: \quad z(1-z)D^2 F + (\gamma-(\alpha+\beta+1)z)DF - \alpha\beta F = 0.$$

After the exercice of previous section, it should be fuchsian. It is actually obvious that equation $HG_{\alpha,\beta,\gamma}$ is meromorphic on $\mathbf{S}$, that its only singularities in $\mathbf{C}$ are 0 and 1 and that they are regular singularities. We shall verify that ∞ is also a regular singularity. We shall also look for the local solutions at singularities, applying the method of Fuchs-Frobenius that we saw in section 8.6. In order to describe the local monodromies, we shall use the base point $1/2$ and the loops described at the end of section 9.1.

Study at 0. We use the first form $HG_{\alpha,\beta,\gamma}$. The indicial equation is $x^2 + (\gamma-1)x = 0$ and the two exponents $0, 1-\gamma$ are non resonant since we assumed that $\gamma \not\in \mathbf{Z}$. Therefore, there is a unique power series solution with constant term 1: it is clearly the hypergeometric series $F(\alpha,\beta,\gamma;z)$ itself; and a unique solution $z^{1-\gamma}G$, where G is a power series solution with constant term 1. To find G, remember that $\delta.z^{1-\gamma} = z^{1-\gamma}.(\delta+1-\gamma)$, so G is solution of the equation:

$$(1-z)(\delta+1-\gamma)^2 G + ((\gamma-1)-(\alpha+\beta)z)(\delta+1-\gamma)G - \alpha\beta z G = 0.$$

Expanding and simplifying, we find:

$$(1-z)\delta^2 G + ((1-\gamma)-(\alpha+\beta+2-2\gamma)z)\delta G - (\alpha+1-\gamma)(\beta+1-\gamma)zG = 0.$$

This is just the hypergeometric equation $HG_{\alpha+1-\gamma,\beta+1-\gamma,2-\gamma}$. Its parameters $\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma$ satisfy the same nonresonancy condition as α,β,γ, so that $G = F(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma;z)$.

Proposition 9.3.1 *A basis of solutions near* 0 *is:*

$$\mathcal{B}_0 := \left(F(\alpha,\beta,\gamma;z), z^{1-\gamma}F(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma;z)\right).$$

$\square$

Corollary 9.3.2 *The monodromy matrix along the loop λ_0 relative to the basis $\mathcal{B}_0$ is $M_0 := \begin{pmatrix} 1 & 0 \\ 0 & e^{-2i\pi\gamma} \end{pmatrix}$.*

Exercice 9.3.3 Describe the two solutions when $\beta = \gamma$.

Study at 1. We must use a local coordinate, that vanishes at $z = 1$. We take $v := 1 - z$. Therefore, if $F(z) = G(v)$, then $F'(z) = -G'(v)$ and $F''(z) = G''(v)$. In symbolic notation, $D_v = -D_z$ and $D_v^2 = D_z^2$. The second form $HG'_{\alpha,\beta,\gamma}$ gives for $G(v)$ the equation:

$$v(1-v)D_v^2 G + ((\alpha+\beta+1-\gamma) - (\alpha+\beta+1)v)D_v G - \alpha\beta G = 0.$$

We recognize the hypergeometric equation $HG'_{\alpha,\beta,\alpha+\beta+1-\gamma}$ with parameters $\alpha, \beta, \alpha+\beta+1-\gamma$. Again for these new parameters, the non resonancy conditions are met. We conclude:

Proposition 9.3.4 *A basis of solutions near 1 is:*

$$\mathcal{B}_1 := \Big(F(\alpha,\beta,\alpha+\beta+1-\gamma;1-z),(1-z)^{\gamma-\alpha-\beta}F(\gamma-\beta,\gamma-\alpha,\gamma+1-\alpha-\beta;1-z)\Big).$$

$\square$

Corollary 9.3.5 *The monodromy matrix along the loop λ_1 relative to the basis $\mathcal{B}_1$ is $M_1 := \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\pi(\gamma-\alpha-\beta)} \end{pmatrix}$.*

Study at ∞. We use the coordinate $w = 1/z$. If $F(z) = G(w)$, then $zF'(z) = -wG'(w)$ which we write symbolically $\delta_w = -\delta_z$. Likewise, $\delta_w^2 = \delta_z^2$. The equation $HG_{\alpha,\beta,\gamma}$ gives for $G(w)$ the equation:

$$(1-\frac{1}{w})\delta_w^2 G - ((\gamma-1) - \frac{\alpha+\beta}{w})\delta_w G - \frac{\alpha\beta}{w}G = 0 \Longleftrightarrow (1-w)\delta_w^2 G - ((\alpha+\beta)-(\gamma-1)w)\delta_w G + \alpha\beta G = 0.$$

This is not an hypergeometric equation (miracles are not permanent !) but it is regular singular, with indicial equation $x^2 - (\alpha+\beta)x + \alpha\beta = 0$. The exponents are α and β and they are non resonant. There fore there are solutions of the form $w^\alpha H_1$ and $w^\beta H_2$, with H_1 and H_2 two power series with constant term 1, and they form a basis. To compute a solution $w^\alpha H$, we apply the rule $\delta_w . w^\alpha = w^\alpha(\delta_w + \alpha)$, whence the equation:

$$(1-w)(\delta_w+\alpha)^2 H - ((\alpha+\beta)-(\gamma-1)w)(\delta_w+\alpha)H + \alpha\beta H = 0 \Longleftrightarrow (1-w)\delta_w^2 H + ((\alpha-\beta)-(2\alpha-\gamma+1)w)\delta_w H - \alpha(\alpha-\gamma-$$

We recognize the hypergeometric equation $HG_{\alpha,\alpha-\gamma+1,\alpha-\beta+1}$ with coefficients $\alpha, \alpha-\gamma+1, \alpha-\beta+1$ and no resonancy, so $H = F(\alpha,\alpha-\gamma+1,\alpha-\beta+1;w)$. The calculation for a solution of the form $z^\beta H$ is symmetric and we conclude:

Proposition 9.3.6 *A basis of solutions near ∞ is:*

$$\mathcal{B}_\infty := \Big((1/z)^\alpha F(\alpha, \alpha-\gamma+1, \alpha-\beta+1; 1/z), (1/z)^\beta F(\beta, \beta-\gamma+1, \beta-\alpha+1; 1/z)\Big).$$

□

To define the local monodromy at ∞, we consider the loop $\lambda_\infty : t \mapsto$

Corollary 9.3.7 *The monodromy matrix along the loop λ_∞ relative to the basis $\mathcal{B}_\infty$ is $M_\infty := \begin{pmatrix} e^{2i\pi\alpha} & 0 \\ 0 & e^{2i\pi\beta} \end{pmatrix}$.*

9.4 Global monodromy according to Riemann

By elementary but genial considerations, Riemann succeeded in finding *explicit* generators of the monodromy group of the hypergeometric equation, but relative to a *non explicit* basis. Starting from the bases $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_\infty$ found above, he considered transformed bases $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_\infty$ whose elements are constant multiples of those of $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_\infty$ but with unspecified coefficients. For instance:

$$\mathcal{C}_0 = \big(p_0 F(\alpha, \beta, \gamma; z), q_0 z^{1-\gamma} F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma; z)\big) = \mathcal{B}_0 D_0, \text{ where } D_0 = \begin{pmatrix} p_0 & 0 \\ 0 & q_0 \end{pmatrix},$$

$p_0, q_0 \in \mathbf{C}^*$ being unspecified, and similarly for $\mathcal{C}_1$ and $\mathcal{C}_\infty$.

We shall consider all functions as defined in the cut plane:

$$\Omega := \mathbf{S} \setminus ([\infty, 0] \cup [1, \infty]) = \mathbf{C} \setminus (]-\infty, 0] \cup [1, +\infty[).$$

This is a simply connected set, so indeed all three bases of germs $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_\infty$ extend to bases of the solution space $\mathcal{F}(\Omega)$. (We use the principal determinations of all z^μ and $(1-z)^\nu$.)

The *connection formulas* are the linear formulas relating the various bases of this space. We write them for $\mathcal{C}_0 = (F_0, G_0)$, $\mathcal{C}_1 = (F_1, G_1)$ and $\mathcal{C}_\infty = (F_\infty, G_\infty)$ in the following form:

$$F_0 = a_1 F_1 + b_1 G_1 = a_\infty F_\infty + b_\infty G_\infty,$$

$$G_0 = c_1 F_1 + d_1 G_1 = c_\infty F_\infty + d_\infty G_\infty.$$

In matricial terms:

$$\mathcal{C}_0 = \mathcal{C}_1 P_1 = \mathcal{C}_\infty P_\infty, \text{ where } P_1 = \begin{pmatrix} a_1 & c_1 \\ b_1 & d_1 \end{pmatrix} \text{ and } P_\infty = \begin{pmatrix} a_\infty & c_\infty \\ b_\infty & d_\infty \end{pmatrix}.$$

The local monodromies were previously found relatively to the bases $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_\infty$:

$$\mathcal{B}_0^{\lambda_0} = \mathcal{B}_0 M_0, \text{ with } M_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2i\pi\gamma} \end{pmatrix},$$

$$\mathcal{B}_1^{\lambda_1} = \mathcal{B}_1 M_1, \text{ with } M_1 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\pi(\gamma-\alpha-\beta)} \end{pmatrix},$$

$$\mathcal{B}_\infty^{\lambda_\infty} = \mathcal{B}_\infty M_\infty, \text{ with } M_\infty = \begin{pmatrix} e^{2i\pi\alpha} & 0 \\ 0 & e^{2i\pi\beta} \end{pmatrix}.$$

All these bases are bases of eigenvectors of the corresponding monodromy matrices. This comes essentially from the fact that matrix D_0 relating $\mathcal{B}_0$ to $\mathcal{C}_0$ commutes with M_0, and similarly at 1 and ∞. Therefore, we also have the local monodromies relative to the bases $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_\infty$:

$$\mathcal{C}_0^{\lambda_0} = \mathcal{C}_0 M_0, \quad \mathcal{C}_1^{\lambda_1} = \mathcal{C}_1 M_1, \quad \text{and} \quad \mathcal{C}_\infty^{\lambda_\infty} = \mathcal{C}_\infty M_\infty.$$

If we can determine P_1 and P_∞, then we will be able to describe the monodromy group relative to $\mathcal{C}_0$. Indeed, we already know that $\mathcal{C}_0^{\lambda_0} = \mathcal{C}_0 M_0$, and also:

$$\mathcal{C}_0^{\lambda_1} = (\mathcal{C}_1 P_1)^{\lambda_1} = \mathcal{C}_1^{\lambda_1} P_1 = \mathcal{C}_1 M_1 P_1 = \mathcal{C}_0 (P_1^{-1} M_1 P_1),$$

$$\mathcal{C}_0^{\lambda_\infty} = (\mathcal{C}_\infty P_\infty)^{\lambda_\infty} = \mathcal{C}_\infty^{\lambda_\infty} P_\infty = \mathcal{C}_\infty M_\infty P_\infty = \mathcal{C}_0 (P_\infty^{-1} M_\infty P_\infty).$$

Therefore, the monodromy group relative to $\mathcal{C}_0$ will be generated by M_0, $P_1^{-1} M_1 P_1$ and $P_\infty^{-1} M_\infty P_\infty$. Moreover, from the relation $\lambda_\infty = (\lambda_0.\lambda_1)^{-1}$, we draw:

$$(P_\infty^{-1} M_\infty P_\infty)(P_1^{-1} M_1 P_1) M_0 = I_2.$$

In the next section, we shall find an explicit connection matrix relating the explicit bases $\mathcal{B}_0$ and $\mathcal{B}_\infty$. Here, we only find connection matrices for the non explicit bases $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_\infty$. The consequence is that one cannot compute the monodromy for a given solution, because we cannot express it in those bases !

Calculation of the connection formulas. We perform analytic continuation along the loop $\lambda_1 = \lambda_0^{-1} \lambda_\infty^{-1}$, applied to the connection formula. For instance, in the first connection formula:

$$F_0 = a_1 F_1 + b_1 G_1 = a_\infty F_\infty + b_\infty G_\infty,$$

the middle expression $a_1 F_1 + b_1 G_1$ is transformed into $a_1 F_1 + b_1 e^{2i\pi(\gamma-\alpha-\beta)} G_1$ along λ_1. On the other hand, $F_0 = a_\infty F_\infty + b_\infty G_\infty$ is left invariant along λ_0^{-1}, then transformed into $a_\infty e^{-2i\pi\alpha} F_\infty + b_\infty e^{-2i\pi\beta} G_\infty$ along λ_∞^{-1}. We eventually get the "new formula":

$$a_1 F_1 + b_1 e^{2i\pi(\gamma-\alpha-\beta)} G_1 = a_\infty e^{-2i\pi\alpha} F_\infty + b_\infty e^{-2i\pi\beta} G_\infty.$$

In the same way, starting from the second connection formula:

$$G_0 = c_1 F_1 + d_1 G_1 = c_\infty F_\infty + d_\infty G_\infty,$$

and noticing that G_0 is multiplied by $e^{2i\pi\gamma}$ along λ_0^{-1}, we get the "new formula":

$$c_1 F_1 + d_1 e^{2i\pi(\gamma-\alpha-\beta)} G_1 = e^{2i\pi\gamma} \left(c_\infty e^{-2i\pi\alpha} F_\infty + d_\infty e^{-2i\pi\beta} G_\infty \right).$$

Now we take some arbitrary $\sigma \in \mathbf{C}$. It will be specialized later to particular values. For each of the two pairs of formulas above (original connection formula and deduced "new formula"), we compute $e^{\sigma i\pi}$ times the original formula minus $e^{-\sigma i\pi}$ times the new formula. Taking in account the general equality:

$$e^{\sigma i\pi} - e^{-\sigma i\pi} \times e^{2\tau i\pi} = 2i \sin(\sigma - \tau)\pi e^{\tau i\pi},$$

we end up with the two following relations:

$$a_1 \sin\sigma\pi F_1 + b_1 \sin(\sigma - (\gamma-\alpha-\beta))\pi e^{(\gamma-\alpha-\beta)i\pi} G_1 = a_\infty \sin(\sigma+\alpha)\pi\, e^{-\alpha i\pi} F_\infty + b_\infty \sin(\sigma+\beta)\pi\, e^{-\beta i\pi} G_\infty,$$
$$c_1 \sin\sigma\pi F_1 + d_1 \sin(\sigma - (\gamma-\alpha-\beta))\pi e^{(\gamma-\alpha-\beta)i\pi} G_1 = c_\infty \sin(\sigma+\alpha-\gamma)\pi\, e^{-(\alpha-\gamma)i\pi} F_\infty + d_\infty \sin(\sigma+\beta-\gamma)\pi\, e^{-(\beta-\gamma)i\pi} G_\infty.$$

If we take $\sigma := \gamma-\alpha-\beta$ in each of these two equalities, we find:

$$a_1 \sin(\gamma-\alpha-\beta)\pi\, F_1 = a_\infty \sin(\gamma-\beta)\pi\, e^{-\alpha i\pi} F_\infty + b_\infty \sin(\gamma-\alpha)\pi\, e^{-\beta i\pi} F_\infty,$$
$$c_1 \sin(\gamma-\alpha-\beta)\pi\, F_1 = c_\infty \sin(-\beta)\pi\, e^{-(\alpha-\gamma)i\pi} F_\infty + d_\infty \sin(-\alpha)\pi\, e^{-(\beta-\gamma)i\pi} F_\infty.$$

Likewise, if we take $\sigma := 0$ in the same two equalities, we find:

$$-b_1 \sin(\gamma-\alpha-\beta)\pi\, e^{(\gamma-\alpha-\beta)i\pi}\, G_1 = a_\infty \sin\alpha\pi\, e^{-\alpha i\pi} F_\infty + b_\infty \sin\beta\pi\, e^{-\beta i\pi} F_\infty,$$
$$-d_1 \sin(\gamma-\alpha-\beta)\pi\, e^{(\gamma-\alpha-\beta)i\pi}\, G_1 = c_\infty \sin(\alpha-\gamma)\pi\, e^{-(\alpha-\gamma)i\pi} F_\infty + d_\infty \sin(\beta-\gamma)\pi\, e^{-(\beta-\gamma)i\pi} F_\infty.$$

Now we make one more special assumption:
All the connection coefficients $a_1, b_1, c_1, d_1, a_\infty, b_\infty, c_\infty, d_\infty$ are supposed to be non zero. This is of course "generically"; the opposite "degenerate" case will be discussed in the last paragraph of this section.

We then have above two expressions of F_1 in the basis $\mathcal{C}_\infty$, and the same for G_1. Identifying, we get:

$$\frac{a_1}{c_1} = \frac{a_\infty}{c_\infty} \frac{\sin(\gamma-\beta)\pi\, e^{-\alpha i\pi}}{\sin(-\beta)\pi\, e^{-(\alpha-\gamma)i\pi}} = \frac{b_\infty}{d_\infty} \frac{}{\sin(-\alpha)\pi\, e^{-(\beta-\gamma)i\pi}},$$
$$\frac{b_1}{d_1} = \frac{a_\infty}{c_\infty} \frac{\sin\alpha\pi\, e^{-\alpha i\pi}}{\sin(\alpha-\gamma)\pi\, e^{-(\alpha-\gamma)i\pi}} = \frac{b_\infty}{d_\infty} \frac{\sin\beta\pi\, e^{-\beta i\pi}}{\sin(\beta-\gamma)\pi\, e^{-(\beta-\gamma)i\pi}}.$$

Now we remember that all the basis elements $F_0, G_0, F_1, G_1, F_\infty, G_\infty$ are defined up to an arbitrary constant factor. This means that we can fix arbitrarily a_1, b_1, c_1, d_1 and one of the four other coefficients. Among the various possibilities, this was the choice of Riemann in his paper on the hypergeometrical functions:

$$P_1 = \frac{1}{\sin(\gamma-\alpha-\beta)\pi} \begin{pmatrix} \sin(\gamma-\alpha)\pi\, e^{-\gamma i\pi} & \sin\beta\pi \\ \sin\alpha\pi\, e^{-(\alpha+\beta)i\pi} & \sin(\gamma-\beta)\pi\, e^{(\gamma-\alpha-\beta)i\pi} \end{pmatrix},$$
$$P_\infty = \frac{1}{\sin(\beta-\alpha)\pi} \begin{pmatrix} \sin(\gamma-\alpha)\pi & \sin(\beta-\gamma)\pi \\ -\sin\alpha\pi & \sin\beta\pi \end{pmatrix}.$$

Theorem 9.4.1 *The monodromy group of $HG_{\alpha,\beta,\gamma}$ expressed in an adequate basis is generated by the matrices M_0, $P_1^{-1}M_1P_1$ and $P_\infty^{-1}M_\infty P_\infty$. These generators obey the relation:*

$$(P_\infty^{-1}M_\infty P_\infty)(P_1^{-1}M_1P_1)M_0 = I_2.$$

□

Exercice 9.4.2 (i) By inspection, verify that the sum of the six exponents (two at each singularity) is an integer.
(ii) Prove this *a priori* by a monodromy argument.

The meaning of the non degeneracy condition on the connection coefficients. If one of the connection coefficients $a_1, b_1, c_1, d_1, a_\infty, b_\infty, c_\infty, d_\infty$ is 0, that means that there is an element of $\mathcal{B}_0$ which is at the same time (up to a non zero constant factor) element of $\mathcal{B}_1$ or $\mathcal{B}_\infty$. Such a function is an eigenvector for the monodromy along λ_0 and at the same time along λ_1 or λ_∞, and therefore an eigenvector for all the monodromy. On the side of the monodromy representation $\pi_1(\mathbf{S} \setminus \{0,1,\infty\}) \to \mathrm{GL}(\mathcal{F}(\Omega))$, this means that there is a subspace (here the line generated by the eigenvector) which is neither $\{0\}$ nor $\mathcal{F}(\Omega)$ and which is stable under the linear action of the monodromy group. Such a representation is said to be *reducible*[1]. On the side of the equation $HG_{\alpha,\beta,\gamma}$, we have a solution f such that $f^{\lambda_0} = e^{2i\pi\mu_0} f$, where $\mu_0 \in \{0, 1-\gamma\}$ and $f^{\lambda_1} = e^{2i\pi\mu_1} f$, where $\mu_1 \in \{0, \gamma-\alpha-\beta\}$. Then $f = z^{\mu_0}(1-z)^{\mu_1} g$ where g is at the same time uniform and of moderate growth at all singularities, whence meromorphic on $\mathbf{S}$, whence rational. Then $v := \dfrac{Df}{f} = \dfrac{Dg}{g} + \dfrac{\mu_0}{z} + \dfrac{\mu_1}{1-z}$ is itself rational. Dividing the hypergeometrical operator by $D - v$ (the same kind of non commutative euclidian division that we performed in the third step of the proof of theorem 8.5.6), we get an equality:

$$D^2 + \frac{\gamma - (\alpha+\beta+1)z}{z(1-z)} D - \frac{\alpha\beta}{z(1-z)} = (D-u)(D-v),$$

with $u, v \in \mathbf{C}(z)$: that is, the hypergeometrical differential operator is *reducible over* $\mathbf{C}(z)$.

9.5 Global monodromy using Barnes connection formulas

Here, we give the results with incomplete explanations and no justification at all, because thay require some more analysis that we are prepared for. See the books WW and GP for details. The main tool is *Barnes integral representation of the hypergeometric series*:

$$F(\alpha,\beta,\gamma;z) = \frac{1}{2i\pi} \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \int_C \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)} (-z)^s \, ds.$$

The line of integration C is the vertical imaginary line, followed from $-i\infty$ to $+i\infty$, with the following deviations: there must be a detour at the left to avoid $-1+\mathbf{N}$; and there must be two detours at the right to avoid $-\alpha-\mathbf{N}$ and $-\beta-\mathbf{N}$. Using this integral representation, Barnes proved the following connection formulas, from which the monodromy is immediately deduced:

$$\begin{aligned} F(\alpha,\beta,\gamma;z) = {} & \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} (-z)^{-\alpha} F(\alpha, \alpha-\gamma+1, \alpha-\beta+1; 1/z) \\ & + \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} (-z)^{-\beta} F(\beta, \beta-\gamma+1, \beta-\alpha+1; 1/z). \end{aligned}$$

Theorem 9.5.1 *(i) One has $\mathcal{B}_0 = \mathcal{B}_\infty P$, where* $P = \begin{pmatrix} e^{-i\pi\alpha} \dfrac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} & e^{-i\pi\alpha'} \dfrac{\Gamma(\gamma')\Gamma(\beta'-\alpha')}{\Gamma(\beta')\Gamma(\gamma'-\alpha')} \\ e^{-i\pi\beta} \dfrac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} & e^{-i\pi\beta'} \dfrac{\Gamma(\gamma')\Gamma(\alpha'-\beta')}{\Gamma(\alpha')\Gamma(\gamma'-\beta')} \end{pmatrix}$.
Here, we set $\alpha' := \alpha-\gamma+1$, $\beta' := \beta-\gamma+1$ and $\gamma' := 2-\gamma$.
(ii) The monodromy group relative to basis $\mathcal{B}_0$ is generated by M_0 and $P^{-1}M_\infty P$.

□

[1] A complete discussion of this case can be found in the books quoted at the beginning of the chapter.

Chapter 10

The global Riemann-Hilbert correspondance

10.1 The correspondance

Fix $a_1, \ldots, a_m \in \mathbf{C}$ and set $\Sigma := \{a_1, \ldots, a_m, \infty\}$; also choose $a \in \Omega := \mathbf{S} \setminus \Sigma$. Then, to each system $X' = AX$ holomorphic on Ω is attached a well defined monodromy representation $\pi_1(\Omega; a) \to \mathrm{GL}(\mathcal{F}_A, a)$. Up to a choice of a fundamental matricial solution of S_A at a, we also get a matricial representation $\pi_1(\Omega; a) \to \mathrm{GL}_n(\mathbf{C})$, but this one is only defined up to conjugacy.

We shall restrict to systems which are regular singular at each point of Σ. To abreviate, we shall call them RS systems (not specifying Σ, which is fixed for the whole chapter). In particular, for a RS system, A must be meromorphic on the whole of $\mathbf{S}$, thus rational: $A \in \mathrm{Mat}_n(\mathbf{C}(z))$. If $\mathcal{X}$ is a fundamental matricial solution of S_A at a, then it defines a multivalued invertible matrix which has moderate growth in sectors in the neighborhood of every point of Σ. As a consequence:

Lemma 10.1.1 *If $F : A \to B$ is a meromorphic equivalence on Ω between two RS systems, then it is a* rational equivalence*: $F \in GL_n(\mathbf{C}(z))$.*

Proof. - Let $\mathcal{X}, \mathcal{Y}$ be fundamental matricial solutions for A, B. Then $F\mathcal{X} = \mathcal{Y}P$ with $P \in \mathrm{GL}_n(\mathbf{C})$ and $F = \mathcal{Y}P\mathcal{X}^{-1}$ is uniform and has moderate growth near points of Σ, so that it is meromorphic at those points, thus on the whole of $\mathbf{S}$ and therefore rational. □

We are going to consider rational equivalence of RS systems. We proved in section 7.6 (under much more general assumptions) that two equivalent systems have conjugate monodromy representations. Therefore, we have a well defined mapping:

{rational equivalence classes of RS systems} $\longrightarrow$ {conjugacy classes of linear representations of $\pi_1(\Omega; a)$}

This is the Riemann-Hilbert correspondance in its most general form.

Proposition 10.1.2 *The above mapping is injective.*

Proof. - Suppose the RS systems with matrices A and B give rise to conjugate monodromy representations. We must show that they are rationally equivalent. We choose fundamental systems

$\mathcal{X}$ and $\mathcal{Y}$ and write $\mathcal{M}_\lambda, N_\lambda$ the monodromy matrices, so that $\mathcal{X}^\lambda = \mathcal{X} M_\lambda$ and $\mathcal{Y}^\lambda = \mathcal{Y} N_\lambda$ for each loop λ in Ω based at a. The assumption is that there exists $P \in \mathrm{GL}_n(\mathbf{C})$ such that $PM_\lambda = N_\lambda P$ for all λ. We put $F := \mathcal{Y} P \mathcal{X}^{-1}$. Then F is uniform:

$$F^\lambda = \mathcal{Y}^\lambda P (\mathcal{X}^\lambda)^{-1} = \mathcal{Y} N_\lambda P M_\lambda^{-1} \mathcal{X}^{-1} = \mathcal{Y} P \mathcal{X}^{-1} = F.$$

It is meromorphic on Ω and has moderate growth near points of Σ, therefore it is meromorphic on $\mathbf{S}$, thus rational. Last, from $F\mathcal{X} = \mathcal{Y} P$ one concludes as usual that $F[A] = B$. $\square$

10.2 The twenty-first problem of Hilbert

At the International Congress of Mathematicians held in 1900 in Paris, Hilbert stated 23 problems meant to inspire mathematicians for the new century. (And so he did: see the two volumes book "Mathematical developments arising from Hilbert problems" edited by the AMS.) In the twenty-first problem, he asked "to show that there always exists a linear differential equation of the fuchsian class with given singular points and monodromy group". The problem admits various interpretations (systems or equations ? of the first kind or regular singular ? with apparent singularities or only true singularities ?), see the books by Anosov-Bolibruch, Deligne, Yoshida and the book GP as well as the above quoted book on Hilbert problems. Some variants have a positive answer, some have a negative or conditional answer. We are going to sketch a proof of the following:

Theorem 10.2.1 *Any representation $\pi_1(\Omega; a) \to GL_n(\mathbf{C})$ can be realized (up to conjugacy) as the monodromy representation of a system S_A which is of the first kind at $a_1, \ldots, a_m$ and which is regular singular at ∞.*

Corollary 10.2.2 *The mapping defined in the previous section (Riemann-Hilbert correspondance) is bijective.*

Remark 10.2.3 As a consequence, RS differential systems can be classified by purely algebraic objects, the representations of the fundamental group (of which the algebraic description is perfectly known). Note however that in the simplest non trivial case, that is $m = n = 2$, we already get a complicated problem, that of classifying the linear two dimensional representations of a free group on two generators. Generally speaking, the case $m = 2$, n arbitrary, is not well understood. See for instance the paper by Deligne "Le groupe fondamental de la droite projective moins trois points".

Exercice 10.2.4 We consider the following equivalence relation on pairs of matrices of $\mathrm{GL}_n(\mathbf{C})$:

$$(M, N) \sim (M', N') \iff \exists P \in \mathrm{GL}_n(\mathbf{C}) \ : \ M' = PMP^{-1} \text{ and } N' = PNP^{-1}.$$

(i) What is the relevance to the above remark ?
(ii) Try to find a classification similar to that for matrices (such as Jordan form, or invariant factors).

The proof of the theorem will proceed in two main steps: on $\mathbf{C}$, then at infinity.

First step: Resolution on C. Up to a rotation, we may assume that all $\mathrm{Re}(a_k)$ are distinct; and up to reindexing, that $\mathrm{Re}(a_1) < \cdots < \mathrm{Re}(a_m)$. Up to conjugacy of the monodromy representation, we can also change the base point and assume that $\mathrm{Re}(a) < \mathrm{Re}(a_1)$. We then define closed rectangles $R_1, \ldots, R_m$ with vertical and horizontal sides, all having the same vertical coordinates and such that: a belongs to none of the R_k; each a_k belongs to the interior of R_k and belongs to no R_l with $l \neq k$; any two consecutive rectangles overlap, *i.e.* they have common interior points.

We consider the prescribed monodromy representation as defined through monodromy matrices $M_1, \ldots, M_m$, each M_k corresponding to a small positive loop in $\mathbf{C}$ around a_k (and around no other a_l). We first solve the problem separately in a neighborhood of each rectangle by using the local theory of chapter 8: this defines for each k a matrix A_k of the first kind and a fundamental matricial solution $\mathcal{X}_k$ having prescribed monodromy matrix M_k.

Suppose the problem has been solved on a neighborhood of the rectangle $R' := R_1 \cup \cdots \cup R_k$, where $k < m$. Call A' and $\mathcal{X}'$ the corresponding system (of the first kind) and fundamental matrix solution (with monodromy matrices $M_1, \ldots, M_k$). Then $\mathcal{X}'$ and $\mathcal{X}_{k+1}$ are analytic and uniform on some simply connected neighborhood of the rectangle $R' \cap R_{k+1}$. According to Cartan lemma stated herebelow, there exist analytic invertible matrices H' on a neighborhood of R' and H_{k+1} on a neighborhood of R_{k+1} such that $H'\mathcal{X}' = H_{k+1}\mathcal{X}_{k+1}$ on a neighborhood of $R' \cap R_{k+1}$. This implies that $H'[A'] = H_{k+1}[A_{k+1}]$ on the same neighborhood, and so that they can be glued into a matrix of the first kind A'' on a neighborhood of $R'' := R' \cup R_{k+1}$, having as fundamental solution the glueing $\mathcal{X}''$ of $H'\mathcal{X}'$ and of $H_{k+1}\mathcal{X}_{k+1}$, which has monodromy matrices $M_1, \ldots, M_{k+1}$. Iterating, we solve the problem in a neighborhood of the rectangle $R_1 \cup \cdots \cup R_m$.

Theorem 10.2.5 (Cartan's lemma) *Let $K' := [a_1, a_3] + i[b_1, b_2]$ and $K'' := [a_2, a_4] + i[b_1, b_2]$, where $a_1 < a_2 < a_3 < a_4$ and $b_1 < b_2$, so that $K := K' \cap K'' = [a_2, a_3] + i[b_1, b_2]$. Let F be an invertible analytic matrix in a neighborhood of K. Then there exist an invertible analytic matrix F' in a neighborhood of K' and an invertible analytic matrix F'' in a neighborhood of K'' such that $F = F'F''$ on a neighborhood of K.*

For a proof, see Gunning and Rossi, "Analytic functions of several complex variables".

Second step: Taking in account ∞. The problem has now been solved on a neighborhood U_0 of a rectangle R containing $a_1, \ldots, a_m$. Up to a translation, we can assume that this rectangle contains 0. There is an anlytic contour $C \subset U_0$ containing R; this means a simple closed curve $t \mapsto C(t)$ defined by the restriction to $[0, 1]$ of an analytic function. (For the existence of such a contour, see the chapters of the books by Ahlfors, Cartan and Rudin devoted to Riemann mapping theorem.) We call A_0 the matrix of the first kind that solves the problem on U_0 and $\mathcal{X}_0$ a corresponding fundamental matricial solution, having monodromy matrices $M_1, \ldots, M_m$.

We also solve the problem locally at infinity on a neighborhood U_∞ of ∞, with a matrix A_∞ of the first kind and a fundamental matricial solution $\mathcal{X}_\infty$ having as monodromy matrix $M_\infty := (M_1 \cdots M_m)^{-1}$.

Now, by Jordan theorem, $\mathbf{S} \setminus C$ has two connected components each homeomorphic to a disk. (See Dieudonné, "Éléments d'analyse, t I" for a proof.) We call D_0 the connected component containing 0 and D_∞ the connected component containing ∞. Then U_0 is a neighborhood of $\overline{D_0}$. There is a neighborhood U_∞ of $\overline{D_\infty}$ such that $\mathcal{X}_0$ and $\mathcal{X}_\infty$ are analytic and invertible on the neighborhood $U_0 \cap U_\infty$ of C. Now, $\mathcal{X}_0$ and $\mathcal{X}_\infty$ are multivalued, but they have the same monodromy along C, so that $M := \mathcal{X}_\infty \mathcal{X}_0^{-1}$ is uniform in a neighborhood of C. We conclude from Birkhoff's preliminary theorem, stated herebelow, that there exist a neighborhood V_0 of $\overline{D_0}$, a neighborhood V_∞ of $\overline{D_\infty}$, an analytic invertible matrix M_0 on V_0, an analytic matrix M_∞ on V_∞ which is invertible on $V_\infty \setminus \{\infty\}$ (but maybe not at ∞) such that $M_0 \mathcal{X}_0 = M_\infty \mathcal{X}_\infty$ on $V_0 \cap V_\infty$. Glueing $M_0[A_0]$ and $M_\infty[A_\infty]$, we get the desired system with matrix A. Note that, since we are not sure that M_∞ is invertible at ∞, we cannot guarantee that A is of the first kind at ∞.

Theorem 10.2.6 (Preliminary theorem of Birkhoff) *With C, D_0, D_∞ as above, suppose we have an invertible analytic matrix M in a neighborhood of C. Then there exist a neighborhood V_0 of $\overline{D_0}$, a neighborhood V_∞ of $\overline{D_\infty}$, an analytic invertible matrix M_0 on V_0, an analytic matrix M_∞ on V_∞ which is invertible on $V_\infty \setminus \{\infty\}$ (but maybe not at ∞) such that $M_0 = M_\infty M$ on $V_0 \cap V_\infty$.*

For a direct proof (without the theory of vector bundles, but using some functional analysis) see Birkhoff, "The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations", American Acad. Proc. 49, 521-568; Amer. Math. Soc. Bull. (2) 19, 508-509 (1913).

Part III

Differential Galois theory

Chapter 11

Local differential Galois theory

The most characteristic feature of the monodromy action is that it preserves algebraic and differential relations. We ara going to axiomatize this fact and study the group of transformations having that property.

In this chapter and the following, we perform the *local* study. Therefore, equations and systems will have coefficients in the field $K := \mathbf{C}(\{z\})$ of germs of meromorphic functions at 0. This field, equipped with the derivation $D := d/dz$, is a *differential field*. Note that we can (and sometimes will) use equivalently the derivation $\delta := zd/dz$.

Remark 11.0.7 The standard approach in this domain has for a long time been to consider extension of K which are themselves differential fields. However, a different point of view begins to spread, of using differential algebras instead. First of all, in spite of the fact that classical Galois theory accustomed us to the theory of fields, they are less natural in the context of differential equations. For instance, $1/\log$ is not solution of a linear differential equation with coefficients in K. Another reason is that in the allied domain of difference and q-difference equations, there appear differential algebras which are not integral and therefore cannot be embedded in a difference field. Since Birkhoff, we have been trying to give as unified a treatment as possible. So here, I chose to do things with algebras.

11.1 The differential algebra generated by the solutions

In the case of an equation $E_{\underline{a}}$ with coefficients $a_1, \ldots, a_n \in K$, we consider a fundamental system of solutions at some point $z_0 \neq 0$, say $\mathcal{B} = (f_1, \ldots, f_n)$, which is a basis for $\mathcal{F}_{\underline{a},z_0}$. In order to be able to express the preservation of algebraic and differential relations, we have to "close" the space of solutions under multiplication and derivation. This is most easily done in the case of systems. (We come back to equations afterwards.)

Let $A \in \mathrm{Mat}_n(K)$ and let $\mathcal{X}$ be a fundamental matricial solution of S_A at some point $z_0 \neq 0$. We shall write $\mathcal{A}(A, z_0)$ the K-algebra generated by the coefficients $x_{i,j}$ of $\mathcal{X}$. All the elements of $\mathcal{A}(A, z_0)$ are polynomial expressions in all the $x_{i,j}$ with coefficients in K. Thus, if we consider the morphism of K-algebras from $K[T_{1,1}, \ldots, T_{n,n}]$ (polynomials in n^2 indeterminates with coefficients in K) to O_{z_0} defined by $T_{i,j} \mapsto x_{i,j}$, the image of this morphism is $\mathcal{A}(A, z_0)$. The absence of $\mathcal{X}$ in the notation is justified by the fact that, if $\mathcal{Y}$ is another fundamental matricial solution at z_0, then $\mathcal{Y} = \mathcal{X}P$ for some $P \in \mathrm{GL}_n(\mathbf{C})$; therefore, each $y_{i,j} = \sum p_{i,k} x_{k,j}$ belongs to the algebra generated by the $x_{i,j}$, and conversely since P is invertible.

Here are the basic facts about $\mathcal{A}(A, z_0)$:

1. As already noted, it does not depend on the choice of a fundamental matricial solution at z_0.

2. If $A \sim B$ (meromorphic equivalence at 0), then $\mathcal{A}(A, z_0) = \mathcal{A}(B, z_0)$. Indeed, let $F \in \mathrm{GL}_n(K)$ such that $F[A] = B$ and let $\mathcal{X}$ be a fundamental system of solutions of S_A at z_0. Then $\mathcal{Y} := F\mathcal{X}$ is a fundamental system of solutions of S_B at z_0. Now, the relations $y_{i,j} = \sum f_{i,k} x_{k,j}$ and the converse relations (using F^{-1}) show that the $x_{i,j}$ and the $y_{i,j}$ generate the same K-algebra.

3. $\mathcal{A}(A, z_0)$ is a sub-differential algebra of the differential K-algebra O_{z_0}. Indeed, it is by definition a sub-algebra. By Leibniz rule, it is enough to check that D sends the generators of $\mathcal{A}(A, z_0)$ into itself; but this follows from the differential system since $D(x_{i,j}) = \sum a_{i,k} x_{k,j}$.

   Exercice 11.1.1 Show rigorously that we have only to consider the case of generators.

4. If z_1 is another point at which A is defined and γ is a path from z_0 to z_1, then analytic continuation along γ yields an isomorphism of differential K-algebras from $\mathcal{A}(A, z_0)$ to $\mathcal{A}(A, z_1)$. This isomorphism depends only on the homotopy class of γ in an open set avoiding the singularities of A (for instance, in a small enough punctured disk centered at 0).

Examples 11.1.2 1. Let $E_{\underline{a}}$ the equation $f^{(n)} + a_1 f^{(n-1)} + \cdots + a_n f = 0$ with coefficients $a_1, \ldots, a_n \in K$. Let $\mathcal{B} := (f_1, \ldots, f_n)$ a fundamental system of solutions at some point $z_0 \neq 0$. Then the algebra $\mathcal{A}(\underline{a}, z_0) := \mathcal{A}(A_{\underline{a}}, z_0)$ is generated by $f_1, \ldots, f_n$ and their derivatives; it is enough to go up to the $(n-1)^{th}$ derivatives.

2. Let $\alpha \in \mathbf{C}$. Then $zf' = \alpha f$ with $z_0 := 1$ gives $\mathcal{A}(A, z_0) = K[z^\alpha]$. The structure of this K-algebra depends on α in the following way:

   - If $\alpha \in \mathbf{Z}$, then of course $\mathcal{A}(A, z_0) = K$.
   - If $\alpha \in \mathbf{Q} \setminus \mathbf{Z}$, then write $\alpha = p/q$ with $p \in \mathbf{Z}$, $q \in \mathbf{N}^*$ and p, q coprime. Then $\mathcal{A}(A, z_0) = K[z^{1/q}]$, which is a field, an algebraic extension of degree q of K. (It is actually a cyclic Galois extension, with Galois group μ_q, the group of q^{th} roots of unity in $\mathbf{C}$.)
   - If $\alpha \in \mathbf{C} \setminus \mathbf{Q}$, then $\mathcal{A}(A, z_0) = K[z^\alpha]$ and z^α is transcendental over K, that is, the morphism of K-algebras from $K[T]$ to $\mathcal{A}(A, z_0)$ sending T to z^α is an isomorphism. Equivalently: the $(z^\alpha)^k$, $k \in \mathbf{N}$, form a basis of $K[z^\alpha]$.

   Exercice 11.1.3 Prove the second and third assertion. (For the second one, the proof is purely algebraic; for the third one, use monodromy.)

3. Consider the equation $zf'' + f' = 0$, $z_0 := 1$ and set $z_0 := 1$. Then $\mathcal{B} := (1, \log)$ is a fundamental system of solutions. Since $1 \in K$ and $\log' \in K$, we have $\mathcal{A}(\underline{a}, z_0) = K[\log]$ and log is transcendental over K.

   Exercice 11.1.4 Prove it.

4. Consider $\alpha \in \mathbf{C}$ and the equation $(\delta-\alpha)^2 f = 0$. A fundamental system of solutions at $z_0 := 1$ is $\mathcal{B} := (z^\alpha, z^\alpha \log)$. Since $\delta(z^\alpha) = \alpha z^\alpha$ and $\delta(z^\alpha \log z) = \alpha z^\alpha \log z + z^\alpha$, the algebra $K[z^\alpha, z^\alpha \log]$ generated by $\mathcal{B}$ is stable under δ, whence it is stable under D, and $\mathcal{A}(\underline{a}, z_0) = K[z^\alpha, z^\alpha \log]$.

   Exercice 11.1.5 Prove that, if $\alpha \in \mathbf{C} \setminus \mathbf{Q}$, then $z^\alpha, z^\alpha \log$ are algebraically independant over K, that is, the morphism of K-algebras from $K[T_1, T_2]$ to $\mathcal{A}(A, z_0)$ sending T_1 to z^α and T_2 to $z^\alpha \log$ is an isomorphism. Equivalently: the elements $(z^\alpha)^k (z^\alpha \log)^l$, $k, l \in \mathbf{N}$, form a basis of $K[z^\alpha, z^\alpha log]$.

5. Consider the equation $f' + \frac{1}{z^2} f = 0$. A fundamental system of solutions is $(e^{1/z})$, so that $\mathcal{A}(\underline{a}, z_0) = K[e^{1/z}]$. Moreover, $e^{1/z}$ is transcendental over K.

   Exercice 11.1.6 Prove it using the growth rate when $z \to 0$ in $\mathbf{R}_+$.

11.2 The differential Galois group

Let $\mathcal{A}$ be an arbitrary differential K-algebra, *i.e.* a K-algebra equipped with a derivation $D : \mathcal{A} \to \mathcal{A}$ extending that of K. An automorphism for that structure is, by definition, an automorphism σ of K-algebra (that is an automorphism of ring which is at the same time K-linear) such that $D \circ \sigma = \sigma \circ D$. If we write more intuitively f' for $D(f)$, this means that $\sigma(f') = (\sigma(f))'$. Note that an automorphism of K-algebra automatically satisfies $\sigma_{|K} = \mathrm{Id}_K$, that is, $\sigma(f) = f$ for all $f \in K$.

Examples 11.2.1
1. Let $\Omega \in \mathbf{C}$ a domain and $a \in \Omega$. Let $\tilde{O}_a$ the differential algebra of analytic germs at a that admit an analytic continuation along every path in Ω starting at a. Then, for every loop λ in Ω based at a, analytic continuation along λ yields a differential automorphism of $\tilde{O}_a$.
2. Let $A \in \mathrm{Mat}_n(K)$ and let $\mathcal{A} := \mathcal{A}(A, z_0)$. Analytic continuation along a loop based at z_0 (and contained in a punctured disk centered at 0 on which A is analytic) transforms a fundamental matricial solution $\mathcal{X}$ into $\mathcal{X}M$ for some $M \in \mathrm{GL}_n(\mathbf{C})$, therefore it transforms any of the generators $x_{i,j}$ of $\mathcal{A}$ into an element of $\mathcal{A}$ and therefore (because of the preservation of algebraic relations) it sends $\mathcal{A}$ into itself. By considering the inverse loop and matrix, one sees that this is a bijection. By preservation of algebraic and differential relations, it is an automorphism of differential K-algebras.

The following lemma shall make it easier to check that a particular σ is a differential automorphism.

Lemma 11.2.2 *Let $\mathcal{A}$ be a differential K-algebra and let σ a K-algebra automorphism of $\mathcal{A}$. Let $f_1, \ldots, f_n$ be generators of $\mathcal{A}$ as a K-algebra. If $\sigma(f_k') = (\sigma(f_k))'$ for $k = 1, \ldots, n$, Then σ is an automorphism of differential K-algebra.*

Proof. - Suppose that $\sigma(f') = (\sigma(f))'$ and $\sigma(g') = (\sigma(g))'$. Then, for $h := \lambda f + \mu g$, $\lambda, \mu \in K$ one has (by easy calculation) $\sigma(h') = (\sigma(h))'$; and the same is true for $h := fg$. Therefore, the set of those f such that $\sigma(f') = (\sigma(f))'$ is a K-algebra containing $f_1, \ldots, f_n$, therefore it is equal to $\mathcal{A}$. □

In all the following examples, we abreviate $\mathcal{A}$ for $\mathcal{A}(A, z_0)$ or $\mathcal{A}(\underline{a}, z_0)$.

Examples 11.2.3 1. Let $zf' = \alpha f$, $\alpha \in \mathbf{C}$ and set $z_0 := 1$. We distinguish three cases:

- If $\alpha \in \mathbf{Z}$, then $\mathcal{A} = K$ and the only differential automorphism is the identity of K.
- If $\alpha \in \mathbf{Q} \setminus \mathbf{Z}$, $\alpha = p/q$, then we saw that $\mathcal{A} = K[z^{1/q}]$. By standard algebra (for instance, field theory in the book of Lang), the automorphisms of K-algebra of $\mathcal{A}$ are defined by $z^{1/q} \mapsto jz^{1/q}$, where $j \in \mu_q$. Now it is easy to see that all of them are differential automorphisms.

  Exercice 11.2.4 Check it. (Use the lemma.)

- If $\alpha \in \mathbf{C} \setminus \mathbf{Q}$, then we saw that $\mathcal{A} = K[z^\alpha]$ and that z^α is transcendental over K. For every differential automorphism σ of $\mathcal{A}$, $\sigma(z^\alpha)$ must be a non trivial solution of $zf' = \alpha f$, thus $\sigma(z^\alpha) = \lambda z^\alpha$ for some $\lambda \in \mathbf{C}^*$. Conversely, this formula defines a unique automorphism σ of the K-algebra $\mathcal{A}$; and since this σ satisfies the condition $\sigma(f') = (\sigma(f))'$ for the generator z^α, by the lemma, it is a differential automorphism.

2. Let $zf'' + f' = 0$ and $z_0 := 1$, so that $\mathcal{A} = K[\log]$ (and log is transcendental over K). From $\log' = 1/z$ one deduces that $(\sigma(\log))' = 1/z$ for every differential automorphism σ. Thus, $\sigma(\log) = \log + \mu$ for some $\mu \in \mathbf{C}$. Conversely, since log is transcendental over K, this defines a unique automorphism of the K-algebra $\mathcal{A}$; and, since it satisfies the condition $\sigma(f') = (\sigma(f))'$ for the generator log, by the lemma, it is a differential automorphism.

3. Let $(\delta - \alpha)^2 f = 0$ and $z_0 := 1$. Then $\mathcal{A} = K[z^\alpha, z^\alpha \log]$. As in the first example, one must have $\sigma(z^\alpha) = \lambda z^\alpha$ for some $\lambda \in \mathbf{C}^*$. Then, from $\delta(z^\alpha \log) = \alpha z^\alpha \log + z^\alpha$, we see that $f := \sigma(z^\alpha \log)$ must satisfy $\delta(f) = \alpha f + \lambda z^\alpha$, so that $g := f - \lambda z^\alpha \log$ satisfies $\delta(g) = \alpha g$, so that $g = \mu z^\alpha$ for some $\mu \in \mathbf{C}$. Therefore, we find that $\sigma(z^\alpha \log) = \lambda z^\alpha \log + \mu z^\alpha$. For the converse, assume that $\alpha \in \mathbf{C} \setminus \mathbf{Q}$. Then z^α and $z^\alpha \log$ being algebraically independent, for any $(\lambda, \mu) \in \mathbf{C}^* \times \mathbf{C}$, there is a unique automorphism σ of the K-algebra $\mathcal{A}$ such that $\sigma(z^\alpha) = \lambda z^\alpha$ and $\sigma(z^\alpha \log) = \lambda z^\alpha \log + \mu z^\alpha$. Since σ satisfies the condition $\sigma(f') = (\sigma(f))'$ for the generators z^α and $z^\alpha \log$, by the lemma, it is a differential automorphism. Note that it all works as if we had computed $\sigma(\log) = \log + \nu$ and set $\mu = \lambda\nu$ but we could not because $\log \notin \mathcal{A}$.

Exercice 11.2.5 What if $\alpha \in \mathbf{Q}$?

Definition 11.2.6 The *differential Galois group* of S_A at z_0, written $\mathrm{Gal}(A, z_0)$, is the group of all differential automorphisms of $\mathcal{A}(A, z_0)$. The *differential Galois group* of $E_{\underline{a}}$ at z_0, is $\mathrm{Gal}(A_{\underline{a}}, z_0)$.

The basic facts are:

1. The monodromy group is a subgroup of the differential Galois group.

2. If γ is a path from z_0 to z_1 and if we write ψ the corresponding differential isomorphism from $\mathcal{A}(A, z_0)$ to $\mathcal{A}(B, z_0)$, then $\sigma \mapsto \psi \circ \sigma \circ \psi^{-1}$ is an isomorphism from the group $\mathrm{Gal}(A, z_0)$ to the group $\mathrm{Gal}(B, z_0)$.

In the examples below, we call for short G the differential Galois group.

Examples 11.2.7 1. Let $zf' = \alpha f$, $\alpha \in \mathbf{C}$ and set $z_0 := 1$. If $\alpha \in \mathbf{Z}$, then $G = \{\mathrm{Id}\}$. If $\alpha \in \mathbf{Q} \setminus \mathbf{Z}$, $\alpha = p/q$, then $G = \mu_q$. If $\alpha \in \mathbf{C} \setminus \mathbf{Q}$, then $G = \mathbf{C}^*$.

2. Let $zf'' + f' = 0$ and $z_0 := 1$, then $G = \mathbf{C}$.

   Exercice 11.2.8 In the two last examples, explain precisely how an element of G acts, *i.e.* how it is to be considered as an automorphism of $\mathcal{A}$. (You may find inspiration in the following example.)

3. Let $(\delta - \alpha)^2 f = 0$ and $z_0 := 1$. Then G can be identified with $\mathbf{C}^* \times \mathbf{C}$. The element $(\lambda, \mu) \in G$ corresponds to the automorphism σ of the K-algebra $\mathcal{A} = K[z^\alpha, z^\alpha \log]$ defined by $\sigma(z^\alpha) = \lambda z^\alpha$ and $\sigma(z^\alpha \log) = \lambda z^\alpha \log + \mu z^\alpha$. In order to see $\mathbf{C}^* \times \mathbf{C}$ as a group, we must understand how to compose elements. So write $\sigma_{\lambda,\mu}$ the automorphism just defined. Then:

$$\begin{aligned}
\sigma_{\lambda',\mu'} \circ \sigma_{\lambda,\mu}(z^\alpha) &= \sigma_{\lambda',\mu'}(\lambda z^\alpha) \\
&= \lambda' \lambda z^\alpha, \\
\sigma_{\lambda',\mu'} \circ \sigma_{\lambda,\mu}(z^\alpha \log) &= \sigma_{\lambda',\mu'}(\lambda z^\alpha \log + \mu z^\alpha) \\
&= \lambda' \lambda z^\alpha \log + (\lambda \mu' + \lambda' \mu) z^\alpha,
\end{aligned}$$

   so that the group law on $\mathbf{C}^* \times \mathbf{C}$ is:

$$(\lambda', \mu') * (\lambda, \mu) := (\lambda' \lambda, \lambda \mu' + \lambda' \mu).$$

   There is yet another way to understand this group. We notice that σ is totally determined by its $\mathbf{C}$-linear action on the space of solutions. The latter has as basis $\mathcal{B} = (z^\alpha, z^\alpha \log)$ and we find $\sigma(\mathcal{B}) = \mathcal{B}M$, where $M = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix}$. Therefore, G can be identified with the subgroup:

$$\left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix} \mid (\lambda, \mu) \in \mathbf{C}^* \times \mathbf{C} \right\} \subset \mathrm{GL}_2(\mathbf{C}).$$

   The reader can check that the multiplication law coincides with the one found above.

Since the monodromy group is a subgroup of the Galois group, we should find it inside each of the examples we computed. This is indeed so:

Examples 11.2.9 1. In the first example, the monodromy group is generated by the factor $\lambda_0 := e^{2i\pi\alpha}$. If $\alpha \in \mathbf{Q}$, it is equal to the Galois group.

2. In the second example, the monodromy group is generated by the constant $\mu_0 := 2i\pi$.

3. In the third case, the monodromy group is generated by the pair $(\lambda_0, \mu_0) := (e^{2i\pi\alpha}, 2i\pi e^{2i\pi\alpha})$; or, in the matricial realisation, by the matrix:

$$\begin{pmatrix} e^{2i\pi\alpha} & 2i\pi e^{2i\pi\alpha} \\ 0 & e^{2i\pi\alpha} \end{pmatrix} = \begin{pmatrix} e^{2i\pi\alpha} & 0 \\ 0 & e^{2i\pi\alpha} \end{pmatrix} \begin{pmatrix} 1 & 2i\pi \\ 0 & 1 \end{pmatrix}.$$

Exercice 11.2.10 For the equation $f' + z^{-2} f$, prove that the Galois group is $\mathbf{C}^*$ while the monodromy group is trivial.

11.3 The Galois group as a linear algebraic group

Let S_A be a system with coefficients in K, $\mathcal{A} := \mathcal{A}(A, z_0)$ be its algebra of solutions at some point z_0 and $\mathrm{Gal} := \mathrm{Gal}(A, z_0)$ be its differential Galois group. For every fundamental matricial solution $\mathcal{X}$ at z_0 and every $\sigma \in G$, one has:

$$(\sigma(\mathcal{X}))' = \sigma(\mathcal{X}') = \sigma(A\mathcal{X}) = A\sigma(\mathcal{X}),$$

so that $\sigma(\mathcal{X}) = \mathcal{X}M$ for some $M \in \mathrm{GL}_n(\mathbf{C})$. (The matrix $\sigma(\mathcal{X})$ has to be invertible because $\det \sigma(\mathcal{X}) = \sigma(\det \mathcal{X}) \neq 0$.) We write M_σ this matrix, and so we have a map $\sigma \mapsto M_\sigma$ from Gal to $\mathrm{GL}_n(\mathbf{C})$. Of course, we hope it to be a representation. It is indeed a morphism (not an anti-morphism) of groups

$$\mathcal{X}M_{\sigma\tau} = (\sigma\tau)(\mathcal{X}) = \sigma(\tau(\mathcal{X})) = \sigma(\mathcal{X}M_\tau) = \sigma(\mathcal{X})M_\tau = \mathcal{X}M_\sigma M_\tau \Longrightarrow M_{\sigma\tau} = M_\sigma M_\tau.$$

Moreover, it is injective: for if $M_\sigma = I_n$, then $\sigma(\mathcal{X}) = \mathcal{X}$ so that the morphism of algebras σ leaves fixed all the generators of the K-algebra $\mathcal{A}$, so that it is actually the identity of $\mathcal{A}$.

Proposition 11.3.1 *The map $\sigma \mapsto M_\sigma$ realizes an isomorphism of Gal(A, z_0) with a subgroup of $GL_n(\mathbf{C})$, the* matricial differential Galois group relative to the fundamental system $\mathcal{X}$. *This subgroup contains the matricial monodromy group relative to $\mathcal{X}$.*

□

Exercice 11.3.2 How are related the matricial differential Galois groups relative to two different fundamental systems ?

Now, we are going to compare the matricial monodromy and Galois groups for our three favorite examples.

Examples 11.3.3

1. In the first example, $n = 1$, $\mathrm{GL}_1(\mathbf{C}) = \mathbf{C}^*$. The monodromy group is $\mathrm{Mon} = \langle e^{2i\pi\alpha} \rangle$. If $\alpha = p/q$, p, q coprime, then $\mathrm{Gal} = \mathrm{Mon} = \mu_q$. If $\alpha \notin \mathbf{Q}$, then $\mathrm{Gal} = \mathbf{C}^*$.
2. In the second example, $n = 2$, the matricial monodromy and Galois group are:
$$\mathrm{Mon} = \left\{ \begin{pmatrix} 1 & 2i\pi k \\ 0 & 1 \end{pmatrix} \mid k \in \mathbf{Z} \right\} \subset \mathrm{Gal} = \left\{ \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \mid \mu \in \mathbf{C} \right\} \subset \mathrm{GL}_2(\mathbf{C}).$$
3. In the third example, $n = 2$; assuming again $\alpha \notin \mathbf{Q}$, the matricial monodromy and Galois group are:
$$\mathrm{Mon} = \left\{ \begin{pmatrix} e^{2i\pi\alpha k} & 2i\pi k e^{2i\pi\alpha k} \\ 0 & e^{2i\pi\alpha k} \end{pmatrix} \mid k \in \mathbf{Z} \right\} \subset \mathrm{Gal} = \left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix} \mid \lambda \in \mathbf{C}^*, \mu \in \mathbf{C} \right\} \subset \mathrm{GL}_2(\mathbf{C}).$$

The big difference is that the Galois group can in all cases be defined within $\mathrm{GL}_2(\mathbf{C})$ by a set of algebraic equations:

1. In the first example, if $\alpha = p/q$, p, q, then:
$$\forall a \in \mathrm{GL}_1(\mathbf{C})\,,\; a \in \mathrm{Gal} \Longleftrightarrow a^q = 1.$$
If $\alpha \notin \mathbf{Q}$, the set of equations is empty.

2. In the second example:
$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{C}) \text{ , } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gal} \Longleftrightarrow a = d = 1 \text{ and } c = 0.$$
3. In the third example:
$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{C}) \text{ , } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gal} \Longleftrightarrow a = d \text{ and } c = 0.$$

Except in the first very special case of a finite monodromy group, there is no corresponding description for Mon.

Theorem 11.3.4 *The matricial realisation of the differential Galois group of A is a* linear algebraic group, *that is a subgroup of $GL_n(\mathbf{C})$ defined by polynomial equations in the coefficients.*

Proof. - We shall write $\mathcal{A} := K[x_{1,1},\dots,x_{n,n}]$ where the $x_{i,j}$ are the coefficients of a fundamental matricial solution $\mathcal{X}$. The matricial realisation of the differential Galois group of A is the set of matrices $P \in \mathrm{GL}_n(\mathbf{C})$ such that there is a morphism of K-algebras from $\mathcal{A}$ to itself sending $\mathcal{X}$ to $\mathcal{X}P$, that is each generator $x_{i,j}$ to $\sum x_{i,k}p_{k,j}$. Indeed, we then get automatically from the relation $\mathcal{X}' = A\mathcal{X}$:
$$\sigma(\mathcal{X}') = \sigma(A\mathcal{X}) = A\sigma(\mathcal{X}) = A\mathcal{X}P = \mathcal{X}'P = (\sigma(\mathcal{X}))',$$
so that the relation $\sigma(x'_{i,j}) = (\sigma(x_{i,j}))'$ is true for all generators $x_{i,j}$ of $\mathcal{A}$, so after the lemma of section 11.2, such a σ is indeed a differential automorphism.

To say that $x_{i,j} \mapsto y_{i,j} := \sum x_{i,k}p_{k,j}$ comes from a morphism of K-algebras from $\mathcal{A}$ to itself is equivalent to say that, for each polynomial relation $F(x_{1,1},\dots,x_{n,n}) = 0$ with coefficients in K, the corresponding relation for the $y_{i,j}$ holds: $F(y_{1,1},\dots,y_{n,n}) = 0$. We shall express this in a slightly different way. We call $(M_\alpha)_{\alpha\in I}$ the family of all monomials $M_\alpha(x_{1,1},\dots,x_{n,n}) = \prod x_{i,j}^{\alpha_{i,j}}$ in the $x_{i,j}$. Therefore, each index α is a matrix $(\alpha_{i,j})$ of exponents, and the set I is the set of such α. For each $\alpha \in I$, we then call N_α the corresponding monomial with the $x_{i,j}$ replaced the by $y_{i,j}$. It is a polynomial expression in the $x_{i,j}$, and therefore a linear combination of the M_α:
$$N_\alpha(x_{1,1},\dots,x_{n,n}) = M_\alpha(y_{1,1},\dots,y_{n,n}) = \prod y_{i,j}^{\alpha_{i,j}} = \sum_\beta \Phi_{\alpha,\beta}(p_{1,1},\dots,p_{n,n})M_\beta(x_{1,1},\dots,x_{n,n}),$$
where the $\Phi_{\alpha,\beta}(p_{1,1},\dots,p_{n,n})$ are themselves polynomial expressions in the $p_{i,j}$ with constant coefficients (actually, these coefficients are in $\mathbf{N}$).

What we ask for P is that, each time there is a linear relation $\sum\lambda_\alpha M_\alpha = 0$ with coefficients $\lambda_\alpha \in K$, the corresponding relation $\sum\lambda_\alpha N_\alpha = 0$ should hold. Now, the latter relation can be written $\sum\mu_\beta M_\beta = 0$, where $\mu_\beta := \sum_\alpha \lambda_\alpha \Phi_{\alpha,\beta}(p_{1,1},\dots,p_{n,n})$.

Lemma 11.3.5 *Let $E \subset K^{(I)}$ be a subspace of the space of all finitely supported families (λ_α). Then, for a family $(\phi_{\alpha,\beta})$ to have the property:*
$$\forall(\lambda_\alpha) \in K^{(I)},\ \textit{setting } \mu_\beta := \sum_\alpha \lambda_\alpha\phi_{\alpha,\beta},\ \textit{one has } (\lambda_\alpha) \in E \Longrightarrow (\mu_\beta) \in E,$$
is equivalent to a family of K-linear conditions of the form $\sum c_{\alpha,\beta}^{(\gamma)}\phi_{\alpha,\beta} = 0$.

Proof. - Left to you as a nice exercice in "abstract linear algebra". □

So there is a family $\Gamma(\gamma)(p_{1,1},\ldots,p_{n,n})$ of polynomials in the $p_{i,j}$ with coefficients in K such that $P \in \mathrm{Gal}$ is equivalent to $\Gamma^{(\gamma)}(p_{1,1},\ldots,p_{n,n}) = 0$ for all γ. But we want polynomial equations with coefficients in $\mathbf{C}$. Therefore, we expand each $\Gamma^{(\gamma)} = \sum_k \Gamma^{(\gamma)} z^k$ and finally obtain the characterisation:

$$P \in \mathrm{Gal} \Longleftrightarrow \forall \gamma, k\ ,\ \Gamma_k^{(\gamma)}(p_{1,1},\ldots,p_{n,n}) = 0.$$

Here, of course, the $\Gamma_k^{(\gamma)} \in \mathbf{C}[T_{1,1},\ldots,T_{n,n}]$ so we do have polynomial equations with coefficients in $\mathbf{C}$. □

Remark 11.3.6 What we found is an infinite family of polynomial equations. But a polynomial ring $\mathbf{C}[T_{1,1},\ldots,T_{n,n}]$ is "noetherian", so that our family can be reduced to a finite set of polynomial equations. This is *Hilbert's basis theorem*: see the chapter on noetherian rings in the book of Lang, or any good book of algebra.

Chapter 12

The local Schlesinger density theorem

We are going to describe more precisely the Galois group for regular singular systems and prove that its matricial realisation is the smallest algebraic subgroup of $\mathrm{GL}_n(\mathbf{C})$ containing the matricial realisation of the monodromy group. This is *Schlesinger density theorem* (here in its local form). We shall not discuss the possibility of extending this result to the global setting, nor to irregular equations. Maybe another year ...

In this chapter, we consider $A \in \mathrm{Mat}_n(K)$ and suppose that S_A is regular singular. Then we know that $A = F[z^{-1}A_0]$ for some $A_0 \in \mathrm{Mat}_n(\mathbf{C})$, so that $\mathcal{X} := Fz^{A_0}$ is a fundamental matricial system for S_A. The matricial monodromy and Galois groups of S_A computed relatively to $\mathcal{X}$ are *equal* to the matricial and monodromy groups of the system $X' = z^{-1}A_0X$ computed relatively to its fundamental matriciaal solution z^{A_0}. Therefore, we will from start study a differential system $X' = z^{-1}AX$ where $A \in \mathrm{GL}_n(\mathbf{C})$. For the same reason, we can and will assume that A is Jordan form (this is because a conjugation of A is also a gauge transformation of $z^{-1}A$). We shall write Mon and Gal the matricial realisations of the monodromy and Galois group relative to z^A.

12.1 Calculation of the differential Galois group in the semi-simple case

We have here $A = \mathrm{Diag}(\alpha_1,\dots,\alpha_n)$ and $\mathcal{X} = \mathrm{Diag}(z^{\alpha_1},\dots,z^{\alpha_n})$, so that $\mathcal{A} = K[z^{\alpha_1},\dots,z^{\alpha_n}]$. For any differential automorphism σ of $\mathcal{A}$, we deduce from the differential relation $\delta(z^{\alpha_i}) = \alpha_i z^{\alpha_i}$ that $f_i := \sigma(z^{\alpha_i})$ satisfies the same: $\delta(f_i) = \alpha_i f_i$. Therefore, f_i must be a constant multiple of z^{α_i}, with non zero coefficient (since σ is an automorphism): $\sigma(z^{\alpha_i}) = \lambda_i z^{\alpha_i}$ for some $\lambda_i \in \mathbf{C}^*$. Thus the elements of the Galois group are diagonal matrices $\mathrm{Diag}(\lambda_1,\dots,\lambda_n) \in \mathrm{GL}_n(\mathbf{C})$. But which anmong these matrices are "galoisian automorphims" ? Writing $a_j := e^{2i\pi a_j}$ for $j = 1,\dots,n$, we know that $\mathrm{Diag}(a_1,\dots,a_n)$ (along with its powers) will fit, but what else ?

The condition was explained in the course of the proof of theorem 11.3.4, section 11.3: to say that $z^{\alpha_i} \mapsto \lambda_i z^{\alpha_i}$ comes from a morphism of K-algebras from $\mathcal{A}$ to itself is equivalent to say that, for each polynomial relation $P(z^{\alpha_1},\dots,z^{\alpha_n}) = 0$ with coefficients in K, the corresponding relation for the $\lambda_i z^{\alpha_i}$ holds: $P(\lambda_1 z^{\alpha_1},\dots,\lambda_n z^{\alpha_n}) = 0$. So we shall have a closer look at the set of such equations:

$$I := \{P \in K[T_1,\dots,T_n] \mid P(z^{\alpha_1},\dots,z^{\alpha_n}) = 0\}.$$

This is an *ideal* of $K[T_1,\dots,T_n]$, *i.e.* a subgroup such that if $P \in I$ and $Q \in K[T_1,\dots,T_n]$, then $PQ \in I$. To describe it, we shall use the monodromy action of the fundamental loop: $\sigma(z^{\alpha_j}) = a_j z^{\alpha_j}$, where $a_j := e^{2i\pi\alpha_j}$. The monodromy group being contained in the Galois group, we know that this σ is an automorphism of $\mathcal{A}$ (principle of preservation of algebraic identities), so that $P(z^{\alpha_1},\dots,z^{\alpha_n}) = 0 \Rightarrow P(a_1 z^{\alpha_1},\dots,a_n z^{\alpha_n}) = 0$. Calling ϕ the automorphism $P(T_1,\dots,T_n) \mapsto P(a_1T_1,\dots,a_nT_n)$ of the K-algebra $K[T_1,\dots,T_n]$, we can say that the subspace I of the K-linear space $K[T_1,\dots,T_n]$ is stable under ϕ.

Lemma 12.1.1 *As a K-linear space, I is generated by the polynomials $T_1^{k_1}\cdots T_n^{k_n} - z^m T_1^{l_1}\cdots T_n^{l_n}$ such that $k_1,\dots,k_n,l_1,\dots,l_n \in \mathbf{N}$, $m \in \mathbf{Z}$ and $k_1\alpha_1+\cdots+k_n\alpha_n = l_1\alpha_1+\cdots+l_n\alpha_n+m$.*

Proof. - If we replace each T_j by z^{α_j} in such a polynomial, we get 0: therefore these polynomials do belong to I.
To prove the converse, we use linear algebra. This can be done in the infinite dimensional linear space $K[T_1,\dots,T_n]$, where the usual theory works quite well (with some adaptations), but for peace of mind we shall do it in finite dimensional subspaces. So we take $d \in \mathbf{N}$ and define E as the subspace of $K[T_1,\dots,T_n]$ made up of polynomials of total degree $\deg P \le d$. A basis of E is the family of monomials $T_1^{k_1}\cdots T_n^{k_n}$ with $k_1+\cdots+k_n \le d$. Since $\phi(T_1^{k_1}\cdots T_n^{k_n}) = a_1^{k_1}\cdots a_n^{k_n} T_1^{k_1}\cdots T_n^{k_n}$, we see that ϕ is a diagonalisable endomorphism of E. This means that E is the direct sum of its eigenspaces E_λ. Clearly, the monomials $T_1^{k_1}\cdots T_n^{k_n}$ with $k_1+\cdots+k_n \le d$ and such that $a_1^{k_1}\cdots a_n^{k_n} = \lambda$ form a basis of E_λ.
Now we set $F := I \cap E$. This is a subspace of E, which is stable under ϕ, so the restriction of ϕ to F is diagonalisable too. Therefore F is the direct sum of its eigenspaces F_λ. We are going to prove that each $F_\lambda = F \cap E_\lambda$ is generated by polynomials of the form stated in the theorem, and from this the conclusion will follow: for then it will be true of their direct sum F, and by letting d tend to $+\infty$ it will be true of I too.
So let $P := \sum f_{\underline{k}} T^{\underline{k}} \in F_\lambda$, where we write for short $\underline{k} := (k_1,\dots,k_n)$ and $T^{\underline{k}} := T_1^{k_1}\cdots T_n^{k_n}$ and where the coefficients $f_{\underline{k}}$ belong to K. The sum is restricted to multi-indices $\underline{k}$ such that $k_1+\cdots+k_n \le d$ and $a_1^{k_1}\cdots a_n^{k_n} = \lambda$, which can be written $k_1\alpha_1+\cdots+k_n\alpha_n = c+m_{\underline{k}}$, where c is an arbitrary constant such that $e^{2i\pi c} = \lambda$ and where $m_{\underline{k}} \in \mathbf{Z}$. All this expresses the fact that $P \in E_\lambda$. The condition that $P \in F$ means that $P(z^{\alpha_1},\dots,z^{\alpha_n}) = 0$, *i.e.* that $\sum f_{\underline{k}} z^{k_1\alpha_1+\cdots+k_n\alpha_n} = 0$, *i.e.* that $z^c \sum f_{\underline{k}} z^{m_{\underline{k}}} = 0$, whence $\sum f_{\underline{k}} z^{m_{\underline{k}}} = 0$. From this, selecting any particular $\underline{l}$ among the $\underline{k}$ involved:

$$P = \sum f_{\underline{k}} T^{\underline{k}} = \sum f_{\underline{k}} T^{\underline{k}} - \left(\sum f_{\underline{k}}\, z^{m_{\underline{k}}}\right) z^{-m_{\underline{l}}} T^{\underline{l}} = \sum f_{\underline{k}} \left(T^{\underline{k}} - z^{m_{\underline{k}}-m_{\underline{l}}} T^{\underline{l}}\right),$$

and we that each re left to check that $T^{\underline{k}} - z^{m_{\underline{k}}-m_{\underline{l}}} T^{\underline{l}}$ is of the expected form. This follows from the following computation:

$$\left.\begin{aligned} k_1\alpha_1+\cdots+k_n\alpha_n &= c+m_{\underline{k}} \\ l_1\alpha_1+\cdots+l_n\alpha_n &= c+m_{\underline{l}} \end{aligned}\right\} \Longrightarrow k_1\alpha_1+\cdots+k_n\alpha_n = (m_{\underline{k}}-m_{\underline{l}}) + l_1\alpha_1+\cdots+l_n\alpha_n.$$

□

Definition 12.1.2 A *replica* of $(a_1,\dots,a_n) \in (\mathbf{C}^*)^n$ is a $(\lambda_1,\dots,\lambda_n) \in (\mathbf{C}^*)^n$ such that:

$$\forall (k_1,\dots,k_n) \in \mathbf{Z}^n,\ a_1^{k_1}\cdots a_n^{k_n} = 1 \Longrightarrow \lambda_1^{k_1}\cdots\lambda_n^{k_n} = 1.$$

Theorem 12.1.3 *The matricial Galois group Gal consists in all diagonal matrices Diag*$(\lambda_1,\dots,\lambda_n) \in GL_n(\mathbf{C})$ *such that* $(\lambda_1,\dots,\lambda_n)$ *is a replica of* $(a_1,\dots,a_n) = (e^{2i\pi\alpha_1},\dots,e^{2i\pi\alpha_n})$.

Proof. - We saw that $\mathrm{Diag}(\lambda_1,\dots,\lambda_n) \in \mathrm{Gal}$ is equivalent to: $\forall P \in I$, $P(\lambda_1 z^{\alpha_1},\dots,\lambda_n z^{\alpha_n}) = 0$. This in turn is equivalent to the same condition restricted to the generators described in the lemma. (If it is true for these generators, it is true for all their linear combinations.) Now, restricted to the generators, the condition reads:

$$\forall \underline{k},\underline{l} \in \mathbf{N}^n \ , \ \forall m \in \mathbf{Z} \ , \ k_1\alpha_1 + \cdots + k_n\alpha_n = l_1\alpha_1 + \cdots + l_n\alpha_n + m \Longrightarrow \prod_{j=1}^n (\lambda_j z^{\alpha_j})^{k_j} = z^m \prod_{j=1}^n (\lambda_j z^{\alpha_j})^{l_j}.$$

After division of both sides by by $z^{\sum k_j\alpha_j} = z^{\sum l_j\alpha_j + m}$, this in turn reads:

$$\forall \underline{k},\underline{l} \in \mathbf{N}^n \ , \ (k_1 - l_1)\alpha_1 + \cdots + (k_n - l_n)\alpha_n \in \mathbf{Z} \Longrightarrow \prod_{j=1}^n \lambda_j^{k_j} = \prod_{j=1}^n \lambda_j^{l_j}.$$

Last, this can be rewritten as the implication: $\forall \underline{k} \in \mathbf{Z}^n$, $k_1\alpha_1 + \cdots + k_n\alpha_n \in \mathbf{Z} \Longrightarrow \prod_{j=1}^n \lambda_j^{k_j} = 1$. But since $k_1\alpha_1 + \cdots + k_n\alpha_n \in \mathbf{Z}$ is equivalent to $a_1^{k_1} \cdots a_n^{k_n} = 1$, this is the definition of a replica. □

The problem of computing the Galois group is now one in abelian group theory. We introduce:

$$\Gamma := \{\underline{k} \in \mathbf{Z}^n \mid k_1\alpha_1 + \cdots + k_n\alpha_n \in \mathbf{Z}\} = \{\underline{k} \in \mathbf{Z}^n \mid a_1^{k_1} \cdots a_n^{k_n} = 1\}.$$

By the general theory of finitely generated abelian groups (see the book of Lang), we know that the subgroup Γ of $\mathbf{Z}^n$ is freely generated by r elements $\underline{k}^{(1)},\dots,\underline{k}^{(r)}$, where $r \leq n$. So there are exactly r conditions to check that a given $(\lambda_1,\dots,\lambda_n) \in (\mathbf{C}^*)^n$ is a replica of $(a_1,\dots,a_n)$. These are monomial conditions: $\lambda_1^{k_1^{(i)}} \cdots \lambda_n^{k_n^{(i)}} = 1$ for $i = 1,\dots,r$.

Example 12.1.4 Consider the equation $zf' = \alpha f$. Here, $\Gamma = \{k \in \mathbf{Z} \mid a^k = 1\} = \{k \in \mathbf{Z} \mid k\alpha \in \mathbf{Z}\}$, where $a := e^{2i\pi\alpha}$. If $\alpha \notin \mathbf{Q}$, then $\Gamma = \{0\}$ and $\mathrm{Gal} = \mathbf{C}^*$. If $\alpha = p/q$, p,q being coprime, then $\Gamma = q\mathbf{Z}$ and $\mathrm{Gal} = \mu_q$.

Example 12.1.5 We consider the system $\delta X = \mathrm{Diag}(\alpha,\beta)X$. Here, $\Gamma = \{(k,l) \in \mathbf{Z}^2 \mid k\alpha + l\beta \in \mathbf{Z}\}$. There are three possible cases according to whether $r = 0, 1$ or 2.

1. The case $r = 2$ arises when $\alpha,\beta \in \mathbf{Q}$. Then Γ is generated by two non proportional elements (k,l) and (k',l') and we have $\mathrm{Gal} = \{\mathrm{Diag}(\lambda,\mu) \in \mathrm{GL}_2(\mathbf{C}) \mid \lambda^k\mu^l = \lambda^{k'}\mu^{l'} = 1\}$. Note that these equations imply $\lambda^{kl'-k'l} = \mu^{kl'-k'l} = 1$, so that Gal is finite. This is related to the fact that all solutions are algebraic, but we just notice this fact empirically here. More precisely, one can prove (it is a nice exercice in algebra) that Γ is either cyclic or isomorphic to the product of two cyclic groups.

2. The case $r = 0$ arises when $1,\alpha,\beta$ are linearly independent over $\mathbf{Q}$. Then $\Gamma = \{0\}$ and Gal contains all invertible diagonal matrices $\mathrm{Diag}(\lambda,\mu)$.

3. The intermediate case $r = 1$ occurs when the $\mathbf{Q}$-linear space $\mathbf{Q} + \mathbf{Q}\alpha + \mathbf{Q}\beta$ has dimension 2. Then Γ is generated by a pair $(k,l) \neq (0,0)$ and $\mathrm{Gal} = \{\mathrm{Diag}(\lambda,\mu) \in \mathrm{GL}_2(\mathbf{C}) \mid \lambda^k\mu^l = 1\}$. It is (again) a nice exercice in algebra to prove that this group is isomorphic to $\mathbf{C}^* \times \mu_q$, where q is the greatest common divisor of k,l.

Exercice 12.1.6 Find an example of a pair (α,β) for each case above.

12.2 Calculation of the differential Galois group in the general case

We now consider a system $\delta X = AX$, where $A \in \mathrm{Mat}_n(\mathbf{C})$ is in Jordan form. Thus, A is a block-diagonal matrix with k blocks $A_i = \alpha_i I_{m_i} + N_{m_i}$, where we write N_m the nilpotent upper triangular matrix $\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \in \mathrm{Mat}_m(\mathbf{C})$.

Exercice 12.2.1 Compute all powers of N_m.

We have $\exp(A \log z) = \mathrm{Diag}(\exp(A_1 \log z), \dots, \exp(A_k \log z))$ and:

$$\exp(A_i \log z) = \exp(\alpha I_{m_i} \log z) \exp(N_{m_i} \log z) \Longrightarrow z^{A_i} = z^{\alpha_i} \left(I_{m_i} + N_{m_i} \log z + \cdots + \frac{\log^{m_i - 1} z}{(m_i - 1)!} N_{m_i}^{m_i - 1} \right).$$

Therefore, the algebra $\mathcal{A}$ is generated by all the $z^{\alpha_i} (\log z)^l$ for $i = 1, \dots, k$ and $0 \leq l \leq m_i - 1$. If $k = n$ and all $m_i = 1$, there is no log at all, but then we are in the semi-simple case of section 12.1.

For any differential automorphism of $\mathcal{A}$, the same calculation as before show that $\sigma(z^{\alpha_i}) = \lambda_i z^{\alpha_i}$ and, if $m_i \geq 1$, $\sigma(z^{\alpha_i} \log z) = \lambda_i z^{\alpha_i} (\log z + \mu_i)$ for some $\lambda_i \in \mathbf{C}^*$ and $\mu_i \in \mathbf{C}$. (We changed slightly the notation for the constant μ.) We also know that $(\lambda_1, \dots, \lambda_k)$ must be a replica of $(e^{2i\pi\alpha_1}, \dots, e^{2i\pi\alpha_k})$. Now, we have two things to consider. First, what about higher powers of log ? From the relation $(z^{\alpha_i} \log z)^l = (z^{\alpha_i})^{l-1} (z^{\alpha_i} (\log z)^l)$, we draw at once that $\sigma(z^{\alpha_i} (\log z)^l) = \lambda_i z^{\alpha_i} (\log z + \mu_i)^l$ for $0 \leq l \leq m_i - 1$. Second thing: how are related the different μ_i ? From the relation $z^{\alpha_j} (z^{\alpha_i} \log z) = z^{\alpha_i} (z^{\alpha_j} \log z)$, we draw at once that all μ_i are equal to a same $\mu \in \mathbf{C}$.

Therefore, any differential automorphism σ is completely determined by the equations $\sigma(z^{\alpha_i} (\log z)^l) = \lambda_i z^{\alpha_i} (\log z + \mu)^l$, where $(\lambda_1, \dots, \lambda_k)$ is a replica of $(e^{2i\pi\alpha_1}, \dots, e^{2i\pi\alpha_k})$ and where $\mu \in \mathbf{C}$. However, we must still see which choices of $(\lambda_1, \dots, \lambda_k)$ and μ do give a differential automorphism. For this, it is enough to prove that they define a K-algebra automorphism. Indeed, the relation $\sigma(f') = (\sigma(f))'$ will then be satisfied by a family of generators, thus by all elements of $\mathcal{A}$ after the lemma of section 11.2.

By the study of the semi-simple case, we know that, the restriction τ of σ to $R := K[z^{\alpha_1}, \dots, z^{\alpha_k}]$ is an automorphism. In the next lemma, we will show that log is transcendental over R, so that any choice of the image of log allows for an extension of τ to an automorphism of $\mathcal{A}' := R[\log]$. In particular, setting $\log \mapsto \log + \mu$, one extends τ to an automorphism σ' of $\mathcal{A}'$. But $\mathcal{A} \subset \mathcal{A}'$ and σ is the restriction of σ' to $\mathcal{A}$. We have proven (admitting temporarily the lemma):

Proposition 12.2.2 *The differential automorphisms of $\mathcal{A}$ are all the maps of the form $\sigma(z^{\alpha_i} (\log z)^l) = \lambda_i z^{\alpha_i} (\log z + \mu)^l$, where $(\lambda_1, \dots, \lambda_k)$ is a replica of $(e^{2i\pi\alpha_1}, \dots, e^{2i\pi\alpha_k})$ and where $\mu \in \mathbf{C}$.*

□

Lemma 12.2.3 *Let $\alpha_1, \ldots, \alpha_k \in \mathbf{C}$ and set $R := K[z^{\alpha_1}, \ldots, z^{\alpha_k}]$. Then $\log$ is transcendental over the field of quotients L of R. Equivalently: there is no non trivial algebraic relation $f_0 + \cdots + f_m \log^m$, with $f_i \in R$.*

Proof. - We extend δ to a derivation over L by putting $\delta(f/g) := (g\delta(f) - f\delta(g))/g^2$, which is well defined: if $f/g = f_1/g_1$, then both possible formulas give the same result.

First Step. Let $m+1$ the minimal degree of an algebraic equation of $\log$ over L. This means that $1, \ldots, \log^m$ are linearly independent over L, while $\log^{m+1} = f_0 + \cdots + f_m \log^m$, with $f_0, \ldots, f_m \in L$. Applying δ, since $\delta(\log^k) = k \log^{k-1}$, we get:

$$(m+1)\log^m = (\delta(f_0) + f_1) + \cdots + (\delta(f_{m-1}) + f_m)\log^{m-1} + \delta(f_m)\log^m,$$

so that, by the assumption of linear independance, $\delta(f_m) = m+1$. This implies that $f_m - (m+1)\log \in \mathbf{C}$ and then that $\log \in L$.

Second step. Suppose that we have $\log \in L$, that is, $f \log = g$ with $f, g \in R$, $f \neq 0$. We use the fact that the action of the monodromy operator σ on R is semi-simple: $f = \sum f_\lambda$ (finite sum), where $\sigma(f_\lambda) = \lambda f_\lambda$; this is because f is a linear combination with coefficients in K (hence invariant under σ) of monomials in the z^{α_j}. Suppose we have written $f \log = g$ with f as "short" as possible, *i.e.* with as few components f_λ as possible. Then, if λ_0 is one of the λ that do appear, we calculate:

$$\sigma(f)(\log + 2i\pi) = \sigma(f \log) = \sigma(g) \Longrightarrow (\sigma(f) - \lambda_0 f)\log = \sigma(g) - 2i\pi\sigma(f) - \lambda_0 g,$$

a shorter relation, except if it is trivial, in which case $\sigma(f) = \lambda_0 f$, which is therefore the only possibility.

Third step. Suppose that we have $f \log = g$ with $f, g \in R$, $f \neq 0$ and $\sigma(f) = \lambda f$. Then, applying σ and simplifying, we find $\sigma(g) - \lambda g = 2i\pi\lambda f$. But this is impossible if $f \neq 0$, because $\sigma - \lambda$ sends g_λ to 0 and all other g_β to elements of the corresponding eigenspaces. This ends the proof of the lemma and of the proposition. □

Theorem 12.2.4 *Let A be in Jordan form $Diag(A_i, \ldots, A_k)$, where $A_i = \alpha_i I_{m_i} + N_{m_i}$. The matricial Galois group Gal of the system $X' = z^{-1}AX$ relatively to the fundamental matricial solution z^A is the set of matrices $Diag(\lambda_1 e^{\mu N_1}, \ldots, \lambda_k e^{\mu N_k})$ where $(\lambda_1, \ldots, \lambda_k)$ is a replica of $(e^{2i\pi\alpha_1}, \ldots, e^{2i\pi\alpha_k})$ and where $\mu \in \mathbf{C}$.*

Proof. - The automorphism σ described in the proposition transforms z^{α_j} into $\lambda_j z^{\alpha_j}$. It transforms $N_j \log z$ into $N_j(\log z + \mu)$ and, because the exponential of a nilpotent matrix is really a polynomial in this matrix, it transforms $e^{N_j \log z}$ into $e^{N_j(\log z + \mu)}$. Therefore, it transforms $z^A = \mathrm{Diag}(z^{\alpha_1} e^{N_1 \log z}, \ldots, z^{\alpha_k} e^{N_k \log z})$. into:

$$\mathrm{Diag}(\lambda_1 z^{\alpha_1} e^{N_1(\log z + \mu)}, \ldots, \lambda_k z^{\alpha_k} e^{N_k(\log z + \mu)}) = z^A M,$$

where $M = \mathrm{Diag}(\lambda_1 e^{\mu N_1}, \ldots, \lambda_k e^{\mu N_k})$. □

Corollary 12.2.5 *The matricial Galois group of the system $X' = F[z^{-1}A]X$ relatively to the fundamental matricial solution Fz^A is the same.*

Exercice 12.2.6 Write down explicitly $e^{\mu N_j}$ and find a set of equations defining Gal in $\mathrm{GL}_n(\mathbf{C})$.

12.3 The density theorem of Schlesinger in the local setting

We suppose that a fundamental matricial solution $\mathcal{X}$ has been chosen for the differential system S_A, so that we have matricial realisations $\mathrm{Mon}(A) \subset \mathrm{Gal}(A) \subset \mathrm{GL}_n(\mathbf{C})$ relative to $\mathcal{X}$.

Theorem 12.3.1 (Local Schlesinger density theorem) *Let the system S_A be regular singular. Then Gal(A) is the smallest algebraic subgroup of GL$_n(\mathbf{C})$ containing Mon(A).*

Proof. - A meromorphic gauge transformation $B = F[A]$, $F \in \mathrm{GL}_n(K)$, gives rise to a fundamental matricial solution $\mathcal{Y}$ such that $F\mathcal{X} = \mathcal{Y}P$, $P \in \mathrm{GL}_n(\mathbf{C})$; then, using matricial realisations relative to $\mathcal{Y}$ of the monodromy and Galois groups of S_B, one has $\mathrm{Mon}(B) = P\mathrm{Mon}(A)P^{-1}$ and $\mathrm{Gal}(B) = P\mathrm{Gal}(A)P^{-1}$. From that, we deduce at the same time that the statement to be proved is independent from the choice of a particular fundamental matricial solution and also that it is invariant up to meromorphic equivalence. Therefore, we take the system in the form $\delta X = AX$, where $A \in \mathrm{Mat}_n(\mathbf{C})$ is in Jordan form. We keep the notations of section 12.2. Therefore, Mon is generated by the monodromy matrix:

$$\begin{aligned}
e^{2i\pi A} &= \mathrm{Diag}(a_1 e^{2i\pi N_1}, \ldots, a_k e^{2i\pi N_k}) \\
&= e^{2i\pi A_s} e^{2i\pi A_n} \qquad \text{(Jordan decomposition)} \\
\text{where } e^{2i\pi A_s} &= \mathrm{Diag}(a_1 I_{m_1}, \ldots, a_k I_{m_k}) \\
\text{and } e^{2i\pi A_n} &= \mathrm{Diag}(e^{2i\pi N_1}, \ldots, e^{2i\pi N_k}).
\end{aligned}$$

Remember that the Jordan decomposition of an invertible matrix into its semi-simple and unipotent component was defined in the corresponding paragraph of section 4.4. Likewise, Gal is the set of matrices of the form:

$$\begin{aligned}
E(\underline{\lambda}, \mu) &:= \mathrm{Diag}(\lambda_1 e^{\mu N_1}, \ldots, \lambda_k e^{\mu N_k}), \\
&= E_s(\underline{\lambda}) E_u(\mu) \qquad \text{(Jordan decomposition)} \\
\text{where } E_s(\underline{\lambda}) &= \mathrm{Diag}(\lambda_1 I_{m_1}, \ldots, \lambda_k I_{m_k}) \\
\text{and } E_u(\mu) &= \mathrm{Diag}(e^{\mu N_1}, \ldots, e^{\mu N_k}).
\end{aligned}$$

where $(\lambda_1, \ldots, \lambda_k) \in (\mathbf{C}^*)^k$ is a replica of $(a_1, \ldots, a_k)$ and where $\mu \in \mathbf{C}$ is arbitrary.

We are supposed to prove that, if $G \subset \mathrm{GL}_n(\mathbf{C})$ is an algebraic subgroup and if $e^{2i\pi A} \in G$, then all matrices of the form $E(\underline{\lambda}, \mu)$ above belong to G. We shall admit the following fact, a proof of which may be found in the book of Borel “Linear algebraic groups”:

Proposition 12.3.2 *If $G \subset GL_n(\mathbf{C})$ is an algebraic subgroup, then, for each $M \in G$, the semi-simple and unipotent components M_s and M_u belong to G.*

Therefore, if G is an algebraic subgroup of $\mathrm{GL}_n(\mathbf{C})$ containing $e^{2\mathrm{i}\pi A}$, then it contains $e^{2\mathrm{i}\pi A_s}$ and $e^{2\mathrm{i}\pi A_n}$. The conclusion of the theorem will therefore follow immediately from the following two lemmas.

Lemma 12.3.3 *If an algebraic subgroup G of $GL_n(\mathbf{C})$ contains $Diag(a_1 I_{m_1}, \ldots, a_k I_{m_k})$, then it contains all matrices of the form $Diag(\lambda_1 I_{m_1}, \ldots, \lambda_k I_{m_k})$ where $(\lambda_1, \ldots, \lambda_k) \in (\mathbf{C}^*)^k$ is a replica of $(a_1, \ldots, a_k)$.*

Proof. - Let $F(T_{1,1}, \ldots, T_{n,n}) \in \mathbf{C}[T_{1,1}, \ldots, T_{n,n}]$ be one of the defining equations of the algebraic subgroup G. We must prove that it vanishes on all matrices of the indicated form. If one replaces the indeterminates $T_{i,j}$ by the corresponding coefficients of the matrix $\mathrm{Diag}(T_1 I_{m_1}, \ldots, T_k I_{m_k})$, one obtains a polynomial $\Phi(T_1, \ldots, T_k) \in \mathbf{C}[T_1, \ldots, T_k]$ such that $\Phi(a_1^p, \ldots, a_k^p) = 0$ for all $p \in \mathbf{Z}$ (because G, being a group, contains all powers of $\mathrm{Diag}(a_1 I_{m_1}, \ldots, a_k I_{m_k})$) and one wants to prove that $\Phi(\lambda_1, \ldots, \lambda_k) = 0$ for all replicas $(\lambda_1, \ldots, \lambda_k)$.
We write Φ as a linear combination of monomials: $\Phi = \sum \lambda_i M_i$. Then, $M_i(a_1^p, \ldots, a_k^p) = M_i(a_1, \ldots, a_k)^p$. We group the indices by packs $I(c)$ such that $M_i(a_1, \ldots, a_k) = c$ for all $i \in I(c)$. Then, if $\Lambda(c) := \sum\limits_{i \in I(c)} \lambda_i$, we see that:

$$\forall p \in \mathbf{Z}\,,\ \Phi(a_1^p, \ldots, a_k^p) = \sum_c \Lambda(c) c^p = 0.$$

By classical properties of the Vandermonde determinant, this implies that $\Lambda(c) = 0$ for every c:

$$\forall c \in \mathbf{C}^*\,,\ \sum_{i \in I(c)} \lambda_i = 0.$$

For every relevant c (*i.e.* such that $I(c)$ is not empty), choose a particular $i_0 \in I(c)$. Then, $\Phi = \sum\limits_c \Phi_c$, where:

$$\Phi_c := \sum_{i \in I(c)} \lambda_i M_i = \sum_{i \in I(c)} \lambda_i (M_i - M_{i_0}).$$

For $i \in I(c)$, one has $M_i(a_1, \ldots, a_k) = M_{i_0}(a_1, \ldots, a_k)$ (both are equal to c); by definition, this monomial relation between the a_i remains true for any replica of $(a_1, \ldots, a_k)$, so that Φ_c vanishes on any replica, and so does Φ. $\Box$

Lemma 12.3.4 *If an algebraic subgroup G of $GL_n(\mathbf{C})$ contains $Diag(e^{2i\pi N_1}, \ldots, e^{2i\pi N_k})$, then it contains all matrices of the form $Diag(e^{\mu N_1}, \ldots, e^{\mu N_k})$ where $\mu \in \mathbf{C}$ is arbitrary.*

Proof. - Let $F(T_{1,1}, \ldots, T_{n,n}) \in \mathbf{C}[T_{1,1}, \ldots, T_{n,n}]$ be one of the defining equations of the algebraic subgroup G. We must prove that it vanishes on all matrices of the indicated form. If one replaces the indeterminates $T_{i,j}$ by the corresponding coefficients of the matrix $\mathrm{Diag}(e^{TN_1}, \ldots, e^{TN_k})$, one obtains a polynomial $\Phi(T) \in \mathbf{C}[T]$: indeed, the matrices N_i being nilpotent, the expressions e^{TN_i} involve only a finite number of powers of T. Moreover, since G is a group, it contains all powers $\mathrm{Diag}(e^{2\mathrm{i}\pi N_1}, \ldots, e^{2\mathrm{i}\pi N_k})^p = \mathrm{Diag}(e^{2\mathrm{i}\pi p N_1}, \ldots, e^{2\mathrm{i}\pi p N_k})$, $p \in \mathbf{Z}$, so that $\Phi(2\mathrm{i}\pi p) = 0$ for all $p \in \mathbf{Z}$. The polynomial Φ has an infinity of roots, it is therefore trivial and $\Phi(\mu) = 0$ for all $\mu \in \mathbf{C}$, which means that $F(T_{1,1}, \ldots, T_{n,n})$ vanishes on all matrices of the indicated form, as wanted. $\Box$

This ends the proof of Schlesinger's theorem. $\Box$

12.4 Why is Schesinger's theorem called a "density theorem" ?

This is going to be a breezy introduction to affine algebraic geometry in the particular case of linear algebraic groups. For every $F(T_{1,1},\ldots,T_{n,n}) \in \mathbf{C}[T_{1,1},\ldots,T_{n,n}]$ and $A = (a_{i,j}) \in \mathrm{Mat}_n(\mathbf{C})$, we shall write for short $F(A) := F(a_{1,1},\ldots,a_{n,n}) \in \mathbf{C}$. (Do not confuse this with matrix polynomials used in reduction theory, such as the minimal and characteristic polynomials of a matrix: here, $F(A)$ is a scalar, not a matrix, and its computation does not involve the powers of A.)

Definition 12.4.1 Let $E \subset \mathbf{C}[T_{1,1},\ldots,T_{n,n}]$ be an arbitrary set of polynomial equations on $\mathrm{Mat}_n(\mathbf{C})$. Then, we write: $V(E) := \{A \in \mathrm{Mat}_n(\mathbf{C}) \mid \forall F \in E\ ,\ F(A) = 0\}$. Such a set is called the *algebraic subset of $Mat_n(\mathbf{C})$ defined by the set of equations E.*

Proposition 12.4.2 *(i) The subsets $\emptyset$ and $Mat_n(\mathbf{C})$ are algebraic subsets.*
(ii) If V_1, V_2 are algebraic subsets, so is $V_1 \cup V_2$.
(ii) If (V_i) is a (possibly infinite) family of algebraic subsets, so is $\bigcap V_i$.

Proof. - (i) One immediately checks that $\emptyset = V(\{1\})$, while $\mathrm{Mat}_n(\mathbf{C}) = V(\{0\})$.
(ii) With a little thought, one finds that $V(E_1) \cup V(E_2) = V(\{F_1.F_2 \mid F_1 \in E_1, F_2 \in E_2\})$.
(iii) One immediately checks that $\bigcap V(E_i) = V(\bigcup E_i)$. $\square$

Corollary 12.4.3 *There is a topology on $Mat_n(\mathbf{C})$ for which the closed subsets are exactly the algebraic subsets.*

We use here the abstract definition of a topology, as a set of open subsets containing $\emptyset$ and $\mathrm{Mat}_n(\mathbf{C})$ and stable under finite intersections and arbitrary unions. Then the closed subsets are defined as the complementary subsets of the open subsets. The topology we just defined is called the Zariski topology. The algebraic subsets are said to be Zariski closed, and the closure $\overline{X}$ of an arbitrary subset X for this topology,called its Zariski closure, is the smallest algebraic subset containing X. Here is a way to "compute" it. Let $I(X) := \{F \in \mathbf{C}[T_{1,1},\ldots,T_{n,n}] \mid \forall A \in X\ ,\ F(A) = 0\}$, the set of all equations satisfied by X. Then $X \subset V(E) \Longleftrightarrow \forall F \in E\ ,\ \forall A \in X\ ,\ F(A) = 0 \Longleftrightarrow E \subset I(X)$. It follows immediately that: $\overline{X} = V(I(X))$.

Another consequence is the following. We say that X is Zariski dense in Y if $X \subset Y \subset \overline{X}$. Then, for a subset X of Y to be Zariski dense, it is necessary and sufficient that the following condition be true: every $F \in \mathbf{C}[T_{1,1},\ldots,T_{n,n}]$ which vanishes on X also vanishes on Y.

Now we consider the restriction of our topology to the open subset $\mathrm{GL}_n(\mathbf{C})$ of $\mathrm{Mat}_n(\mathbf{C})$. (It is open because it is the complementary subset of $V(\det)$ and $\det \in \mathbf{C}[T_{1,1},\ldots,T_{n,n}]$.) Then one can prove that the closure in $\mathrm{GL}_n(\mathbf{C})$ of a subgroup G of $\mathrm{GL}_n(\mathbf{C})$ is a subgroup of $\mathrm{GL}_n(\mathbf{C})$ (see the book of Borel for a proof). Of course, this closure is then exactly what we called an algebraic subgroup and so it is the smallest algebraic subgroup of $\mathrm{GL}_n(\mathbf{C})$ containing G. The translation of Schlesinger theorem in this language is therefore:

Corollary 12.4.4 *The monodromy group Mon is Zariski dense in the algebraic group Gal.*

Exercice 12.4.5 Among the classical subgroups of $\mathrm{GL}_n(\mathbf{C})$, which are Zariski closed ? Which are Zariski dense ?

Chapter 13

Supplementary chapter: the Universal (Fuchsian Local) Galois Group

We take the same notations as in chapter 12. So let $\mathcal{X}$ be a fundamental matricial solution of the system $X' = AX$, with $A \in \mathrm{Mat}_n(K)$. We defined the monodromy representation $\rho_A : \pi_1 \to \mathrm{GL}_n(\mathbf{C})$ by the formula:

$$[\lambda] \mapsto M_\lambda := \mathcal{X}^{-1}\mathcal{X}^\lambda.$$

Then, we defined the monodromy group as:

$$\mathrm{Mon}(A) := \mathrm{Im}\ \rho_A.$$

In the *fuchsian case*, *i.e.* when 0 is a regular singular point of the system $X' = AX$, we obtained a bijective correspondance:

$$\left\{ \begin{matrix} \text{isomorphism classes of} \\ \text{regular singular systems} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{isomorphism classes of} \\ \text{representations of } \pi_1 \end{matrix} \right\}$$

On the side of Galois theory, we first introduced the differential algebra $\mathcal{A} := K[\mathcal{X}]$, with its group of differential automorphisms $\mathrm{Aut}(\mathcal{A})$; then we defined the Galois group through its matricial realisation, as the image of the group morphism $\mathrm{Aut}(\mathcal{A}) \mapsto \mathrm{GL}_n(\mathbf{C})$ defined by the formula:

$$\sigma \mapsto \mathcal{X}^{-1}(\sigma\mathcal{X}).$$

The main difference with the monodromy representation is the following: the source of ρ_A, the group π_1, was independant of the particular system being studied, it was a "universal" group. On the other hand, the group $\mathrm{Aut}(\mathcal{A})$ is obviously related to A, it is by no way universal.

Our goal is here to construct a universal group $\hat{\pi}_1$ and, for each particular system $X' = AX$, a representation $\hat{\rho}_A : \hat{\pi}_1 \to \mathrm{GL}_n(\mathbf{C})$, in such a way that:

- The Galois group is the image of that representation: $\mathrm{Gal}(A) = \mathrm{Im}\ \hat{\rho}_A$.
- The "functor" $A \rightsquigarrow \hat{\rho}_A$ induces a bijective correspondance between isomorphism classes of regular singular systems and isomorphism classes of representations of $\hat{\pi}_1$ ("algebraic Riemann-Hilbert correspondance").

We shall be able to do that for local regular singular systems (although, as in the case of monodromy, more general results are known for global systems, as well as for irregular systems). However, we shall have to take in account the fact that the Galois group is always an algebraic subgroup of $\mathrm{GL}_n(\mathbf{C})$, and therefore restrict the class of possible representations to enforce that property. We begin with two purely algebraic sections.

13.1 Some algebra, with replicas

Remember the definition 12.1.2 of *replicas* in section 12.1 of chapter 12. The following criterion is due to Chevalley.

Theorem 13.1.1 *Let $(a_1,\ldots,a_n) \in (\mathbf{C}^*)^n$. Then $(b_1,\ldots,b_n) \in (\mathbf{C}^*)^n$ is a replica of $(a_1,\ldots,a_n)$ if, and only if, there exists a group morphism $\gamma : \mathbf{C}^* \to \mathbf{C}^*$ such that $\gamma(a_i) = b_i$ for $i = 1,\ldots,n$.*

Proof. - Clearly, if such a morphism γ exists, then, for any $(m_1,\ldots,m_n) \in \mathbf{Z}^n$:

$$a_1^{m_1}\cdots a_n^{m_n} = 1 \Longrightarrow \gamma(a_1^{m_1}\cdots a_n^{m_n}) = 1 \Longrightarrow b_1^{m_1}\cdots b_n^{m_n} = 1,$$

so that $(b_1,\ldots,b_n)$ is indeed a replica of $(a_1,\ldots,a_n)$.
Now assume conversely that $(b_1,\ldots,b_n)$ is a replica of $(a_1,\ldots,a_n)$. For $i = 1,\ldots,n$, let $\Gamma_i := < a_1,\ldots,a_i >$ be the subgroup of $\mathbf{C}^*$ generated by $a_1,\ldots,a_i$. We are going first to construct, for each $i = 1,\ldots,n$, a group morphism $\gamma_i : \Gamma_i \to \mathbf{C}^*$ such that $\gamma_i(a_j) = b_j$ for $j = 1,\ldots,i$; these morphisms will be extensions of each other, *i.e.* $\gamma_{i|\Gamma_{i-1}} = \gamma_{i-1}$ for $i = 2,\ldots,n$.
For $i = 1$, we know that $a^m = 1 \Rightarrow b^m = 1$, so it is an easy exercice in group theory to show that setting $\gamma_1(a^k) := b^k$ makes sense and defines a group morphism $\gamma_1 : \Gamma_1 \to \mathbf{C}^*$.
Suppose that $\gamma_i : \Gamma_i \to \mathbf{C}^*$ has been constructed and that $i < n$. Any element of Γ_{i+1} can be written $g a_{i+1}^k$ for some $g \in \Gamma_i$ and $k \in \mathbf{Z}$; but of course, this decomposition is not necessarily unique ! However:

$$g a_{i+1}^k = g' a_{i+1}^{k'} \Rightarrow g^{-1}g' = a_{i+1}^{k-k'} \Rightarrow \gamma_i(g^{-1}g') = b_{i+1}^{k-k'} \Rightarrow \gamma_i(g) b_{i+1}^k = \gamma_i(g') b_{i+1}^{k'},$$

so that it makes sense to set $\gamma_{i+1}(g a_{i+1}^k) := \gamma_i(g) b_{i+1}^k$ and it is (again) an easy exercice to check that this γ_{i+1} is a group morphism $\Gamma_{i+1} \to \mathbf{C}^*$ extending γ_i.
Therefore, in the end, we have $\gamma_n : \Gamma_n \to \mathbf{C}^*$ such that $\gamma_n(a_i) = b_i$ for $i = 1,\ldots,n$ and it suffices to apply the following lemma with $\Gamma := \mathbf{C}^*$ and $\Gamma' := \Gamma_n$. □

Lemma 13.1.2 *Let $\Gamma' \subset \Gamma$ be abelian groups and let $\gamma' : \Gamma' \to \mathbf{C}^*$ be a group morphism. Then γ' can be extended to Γ,* i.e. *there is a group morphism $\gamma : \Gamma \to \mathbf{C}^*$ such that $\gamma_{|\Gamma'} = \gamma'$.*

Proof. - The first part of the proof relies on a mysterious principle from the theory of sets, called "Zorn's lemma" (see the book of Lang). We consider the set:

$$\mathcal{E} := \{(\Gamma'',\gamma'') \mid \Gamma' \subset \Gamma'' \subset \Gamma \text{ and } \gamma'' : \Gamma'' \to \mathbf{C}^* \text{ and } \gamma''_{|\Gamma'} = \gamma'\},$$

where of course Γ'' runs among subgroups of Γ and γ'' among group morphisms from Γ'' to $\mathbf{C}^*$. We define an *order* on $\mathcal{E}$ by setting:

$$(\Gamma''_1,\gamma''_1) \prec (\Gamma''_2,\gamma''_2) \Longleftrightarrow \Gamma''_1 \subset \Gamma''_2 \text{ and } (\gamma''_2)_{|\Gamma''_1} = \gamma''_1.$$

Then $(\mathcal{E}, \prec)$ is an *inductive ordered set*. This means that for any family $\{(\Gamma''_i, \gamma''_i)_{i\in I}\}$ of elements of $\mathcal{E}$ which is assumed to be *totally ordered*, *i.e.*:

$$\forall i, j \in I \ , \ (\Gamma''_i, \gamma''_i) \prec (\Gamma''_j, \gamma''_j) \text{ or } (\Gamma''_j, \gamma''_j) \prec (\Gamma''_i, \gamma''_i),$$

(such a family is called a *chain*), there is an *upper bound*, *i.e.* an element $(\Gamma'', \gamma'') \in \mathcal{E}$ such that:

$$\forall i \in I \ , \ (\Gamma''_i, \gamma''_i) \prec (\Gamma'', \gamma'').$$

Indeed, we take $\Gamma'' := \bigcup_{i\in I} \Gamma''_i$ and check that this is a group (we have to use the fact that the family of subgroups Γ''_i is totally ordered, *i.e.* for any two of them, one is included in the other); then we define $\gamma'' : \Gamma'' \to \mathbf{C}^*$ such that its restriction to each Γ''_i is γ''_i (we have to use the fact that the family of elements (Γ''_i, γ''_i) is totally ordered, *i.e.* for any two of them, one of the morphisms extends the other).
Now, the ordered set $(\mathcal{E}, \prec)$ being inductive, Zorn's lemma states that it admits a *maximal element* (Γ'', γ''). This means that γ'' extends γ' but that it cannot be extended further. It is now enough to prove that $\Gamma'' = \Gamma$.
So assume by contradiction that there exists $x \in \Gamma \setminus \Gamma''$ and define $\Gamma''' :=< \Gamma'', x >$, the subgroup of Γ generated by Γ'' and x (it contains strictly Γ''). We are going to extend γ'' to a morphism $\gamma''' : \Gamma''' \to \mathbf{C}^*$, thereby contradicting the maximality of (Γ'', γ''). The argument is somewhat similar to the proof of the theorem (compare them !); there are two cases to consider:

1. If $x^N \in \Gamma'' \Rightarrow N = 0$, then any element of Γ''' can be uniquely written gx^k with $g \in \Gamma''$ and $k \in \mathbf{Z}$. In this case, we choose $y \in \mathbf{C}^*$ arbitrary and it is an easy exercice in group theory to check that setting $\gamma'''(gx^k) := \gamma''(g)y^k$ makes sense and meets our requirements.

2. Otherwise, there is a unique $d \in \mathbf{N}^*$ such that $x^d \in \Gamma'' \Leftrightarrow N \in d\mathbf{Z}$ (this is because such exponents N make up a subgroup $d\mathbf{Z}$ of $\mathbf{Z}$). Then we choose $y \in \mathbf{C}^*$ such that $y^d = \gamma''(x^d)$ (the latter is a well defined element of $\mathbf{C}^*$). Now, any element of Γ''' can be *non uniquely* written gx^k with $g \in \Gamma''$ and $k \in \mathbf{Z}$, and, again, it is an easy exercice in group theory to check that setting $\gamma'''(gx^k) := \gamma''(g)y^k$ makes sense and meets our requirements.

□

Remark 13.1.3 In the theorem, the groups $\Gamma, \Gamma', \ldots$ on the left hand side of the morphisms can be arbitrary abelian groups, but this is not true for the group $\mathbf{C}^*$ on the right hand side. The reader can check that the decisive property of $\mathbf{C}^*$ that was used is the fact that it is *divisible*: for all $d \in \mathbf{N}^*$, the map $y \mapsto y^d$ is surjective.

Exercice 13.1.4 (i) For any abelian group Γ, define $X(\Gamma) := \mathrm{Hom}_{gr}(\Gamma, \mathbf{C}^*)$, the set of group morphisms $\Gamma \to \mathbf{C}^*$. Show that defining a product $\star$ on $X(\Gamma)$ by the formula $(\gamma_1 \star \gamma_2)(x) := \gamma_1(x).\gamma_2(x)$ gives $X(\Gamma)$ the structure of an abelian group.
(ii) Show that any group morphism $f : \Gamma_1 \to \Gamma_2$ yields a "dual" morphism $X(f) : \gamma \mapsto \gamma \circ f$ from $X(\Gamma_2)$ to $X(\Gamma_1)$.
(iii) Check that if f is injective (resp. surjective), then $X(f)$ is surjective (resp. injective). (Note that one of these statements is trivial while the other depends on the non trivial lemma above !)
(iv) If $p : \Gamma \to \Gamma''$ is surjective with kernel Γ', show that $X(p)$ induces an isomorphism from $X(\Gamma'')$ to the kernel of the natural (restriction) map $X(\Gamma) \mapsto X(\Gamma')$ and conclude that $X(\Gamma') \simeq X(\Gamma)/X(\Gamma'')$.

13.2 Algebraic groups and replicas of matrices

Let $M \in \mathrm{GL}_n(\mathbf{C})$. (In a moment, we shall take $M := e^{2\mathrm{i}\pi A}$, the fundamental monodromy matrix of the system $X' = z^{-1}AX$, where $A \in \mathrm{GL}_n(\mathbf{C})$.) Let $M = M_s M_u = M_u M_s$ its Jordan decomposition and write $M_s = P\mathrm{Diag}(a_1, \ldots, a_n)P^{-1}$. It follows from chapter 12 that the smallest algebraic group containing M (*i.e.* the Zariski closure $\overline{<M>}$) is the set of matrices $P\mathrm{Diag}(b_1, \ldots, b_n)P^{-1}M_u^{\lambda}$, where $(b_1, \ldots, b_n)$ is a replica of $(a_1, \ldots, a_n)$ and where $\lambda \in \mathbf{C}$. Now, we introduce a new notation; for every $\gamma \in \mathrm{Hom}_{gr}(\mathbf{C}^*, \mathbf{C}^*)$, we put:

$$\gamma(M_s) := P\mathrm{Diag}(\gamma(a_1), \ldots, \gamma(a_n))P^{-1}.$$

It is absolutely not tautological that this makes sense, *i.e.* that the right hand side of the equality depends on M_s only and not on the particular choice of the diagonalisation matrix P: see the exercice 13.2.3. Then the criterion of Chevalley allows us to conclude:

Corollary 13.2.1 *The smallest algebraic group containing M is:*

$$\overline{<M>} = \left\{ \gamma(M_s)M_u^{\lambda} \mid \gamma \in \mathit{Hom}_{gr}(\mathbf{C}^*, \mathbf{C}^*), \lambda \in \mathbf{C} \right\}.$$

□

In terms of representations, it is obvious that $<M>$ is the image of $\rho : \mathbf{Z} \to \mathrm{GL}_n(\mathbf{C}), k \mapsto M^k$ (this abstracts the definition of the monodromy group); but it now follows that $\overline{<M>}$ can also be obtained as the image of some representation:

$$\hat{\rho} : \begin{cases} (\gamma, \lambda) \mapsto \gamma(M_s)M_u^{\lambda}, \\ \hat{\pi}_1 \to \mathrm{GL}_n(\mathbf{C}), \end{cases} \qquad \text{where we put } \hat{\pi}_1 := \mathrm{Hom}_{gr}(\mathbf{C}^*, \mathbf{C}^*) \times \mathbf{C}.$$

(The reader should check that this is indeed a group morphism.) We get the following diagram, where we write π_1 for $\mathbf{Z}$

$$\begin{tikzcd}
\pi_1 \arrow[dr, "\rho"] & \\
& \mathrm{GL}_n(\mathbf{C}) \\
\hat{\pi}_1 \arrow[ur, "\hat{\rho}"'] &
\end{tikzcd}$$

But of course, $<M> \subset \overline{<M>}$, *i.e.* $\mathrm{Im}\,\rho \subset \mathrm{Im}\,\hat{\rho}$, *i.e.* each M^k can be written in the form $\gamma(M_s)M_u^{\lambda}$: and indeed, this is obviously true if we choose $\lambda := k$ and $\gamma : z \mapsto z^k$. Therefore, we complete the above diagram by defining $\iota : \pi_1 \mapsto \hat{\pi}_1$ by the formula:

$$\iota(k) := \left((z \mapsto z^k), k \right) \in \mathrm{Hom}_{gr}(\mathbf{C}^*, \mathbf{C}^*) \times \mathbf{C}.$$

(Note that this injective group morphism identifies $\pi_1 = \mathbf{Z}$ with a subgroup of $\hat{\pi}_1 = \mathrm{Hom}_{gr}(\mathbf{C}^*, \mathbf{C}^*) \times \mathbf{C}$.) In the end, we get the following *commutative* diagram:

$$\begin{tikzcd}
\pi_1 \arrow[d, "\iota"'] \arrow[dr, "\rho"] & \\
& \mathrm{GL}_n(\mathbf{C}) \\
\hat{\pi}_1 \arrow[ur, "\hat{\rho}"'] &
\end{tikzcd}$$

It is clear that every representation $\rho : \pi_1 \to \mathrm{GL}_n(\mathbf{C})$ gives rise to a cyclic[1] subgroup:

$$\mathrm{Im}\, \rho = < \rho(1) >$$

of $\mathrm{GL}_n(\mathbf{C})$; and conversely, every cyclic subgroup og $\mathrm{GL}_n(\mathbf{C})$ can obviously be obtained as the image of such a representation of $\pi_1 = \mathbf{Z}$.

As for the algebraic subgroups of $\mathrm{GL}_n(\mathbf{C})$ of the form $\overline{< M >}$, it follows from the previous discussion that all of them can be obtained as the image of a representation $\hat{\rho} : \hat{\pi}_1 \to \mathrm{GL}_n(\mathbf{C})$. (Just take the one described above.) But *the converse is false*. Not all representations are admissible. The general theory says that $\hat{\pi}_1$ is a "proalgebraic" group and that the admissible representations are the "(pro)-rational" ones; this is explained in rather elementary terms in the course "Représentations des groupes algébriques et équations fonctionnelles", to be found at `http://www.math.univ-toulouse.fr/~sauloy/PAPIERS/dea08-09.pdf`. We shall only illustrate this by two necessary conditions.

Note first that any representation $\hat{\rho}$ of $\hat{\pi}_1 = \mathrm{Hom}_{gr}(\mathbf{C}^*, \mathbf{C}^*) \times \mathbf{C}$ actually has two components: a representation $\hat{\rho}_s$ of $\mathrm{Hom}_{gr}(\mathbf{C}^*, \mathbf{C}^*)$, and a representation $\hat{\rho}_u$ of $\mathbf{C}$. Conversely, given the representations $\hat{\rho}_s$ and $\hat{\rho}_u$, we can recover $\hat{\rho}$ by setting $\hat{\rho}(M) := \hat{\rho}_s(M_s)\hat{\rho}_u(M_u)$; the only necessary condition is that every element of $\mathrm{Im}\, \hat{\rho}_s \subset \mathrm{GL}_n(\mathbf{C})$ commutes with every element of $\mathrm{Im}\, \hat{\rho}_u \subset \mathrm{GL}_n(\mathbf{C})$. So we shall find independant conditions on $\hat{\rho}_s$ and $\hat{\rho}_u$.

It is clear that $\hat{\rho}_u : \mathbf{C} \to \mathrm{GL}_n(\mathbf{C})$ should have the form $\lambda \mapsto U^\lambda$ for some unipotent matrix U. But there are many representations that do not have this form, for instance all maps $\lambda \mapsto e^{\phi(\lambda)} I_n$ where ϕ is any nontrivial morphism from $\mathbf{C}$ to itself.

We now look for a condition on $\hat{\rho}_s$. Let Γ be any finitely generated subgroup of $\mathbf{C}^*$. Then, as shown in exercice 13.1.4, $X(\Gamma)$ can be identified with the quotient of $X(\mathbf{C}^*) = \mathrm{Hom}_{gr}(\mathbf{C}^*, \mathbf{C}^*)$ by the kernel $X(\mathbf{C}^*/\Gamma)$ of the surjective map $X(\mathbf{C}^*) \to X(\Gamma)$. Taking for Γ the subgroup generated by the eigenvalues of M, we see that every admissible representation $\hat{\rho}_s$ must be trivial on the subgroup $X(\mathbf{C}^*/\Gamma)$ of $X(\mathbf{C}^*) = \mathrm{Hom}_{gr}(\mathbf{C}^*, \mathbf{C}^*)$ for some finitely generated subgroup Γ of $\mathbf{C}^*$. (Actually, this is a sufficient condition, as shown in the course quoted above.) Now, the reader will easily construct a morphism from $X(\mathbf{C}^*) = \mathrm{Hom}_{gr}(\mathbf{C}^*, \mathbf{C}^*)$ to $\mathrm{GL}_1(\mathbf{C}) = \mathbf{C}^*$ that is not admissible in this sense. (See exercice 13.2.4.)

[1]Here, "cyclic" means "generated by one element"; in french terminology, this is called "monogène" while "cyclique" is reserved for a *finite* cyclic group, *i.e.* one generated by an element of finite order. In this particular case, the french terminology is more logical (it implies that there are "cycles").

Remark 13.2.2 The relation between the abstract group π_1 and the proalgebraic group $\hat{\pi}_1$ implies that *every representation of* π_1 *is the restriction of a unique "rational" representation of* $\hat{\pi}_1$: one says that $\hat{\pi}_1$ is the "proalgebraic hull" of π_1.

Exercice 13.2.3 If $f : \mathbf{C} \to \mathbf{C}$ is an arbitrary map, if $\lambda_i, \mu_i \in \mathbf{C}$ for $i = 1, \ldots, n$ and if $P, Q \in \mathrm{GL}_n(\mathbf{C})$, then prove the following implication:

$$P\mathrm{Diag}(\lambda_1, \ldots, \lambda_n)P^{-1} = Q\mathrm{Diag}(\mu_1, \ldots, \mu_n)Q^{-1} \Longrightarrow P\mathrm{Diag}(f(\lambda_1), \ldots, f(\lambda_n))P^{-1} = Q\mathrm{Diag}(f(\mu_1), \ldots, f(\mu_n))Q^{-1}.$$

(This remains true when $\mathbf{C}$ is replaced by an arbitrary commutative ring.)

Exercice 13.2.4 Construct a morphism from $X(\mathbf{C}^*) = \mathrm{Hom}_{gr}(\mathbf{C}^*, \mathbf{C}^*)$ to $\mathrm{GL}_1(\mathbf{C}) = \mathbf{C}^*$ that is not admissible in the above sense. (Use exercice 13.1.4.)

13.3 The universal group

The group $\hat{\pi}_1$ has been introduced in order to parameterize all algebraic groups of the form $\overline{<M>}$. Therefore, it also parameterizes all differential Galois groups of local fuchsian systems. We now describe in more detail this application.

We start from the system $X' = z^{-1}AX$, where $A \in \mathrm{GL}_n(\mathbf{C})$; we know that this restriction does not reduce the generality of our results. Let $\mathcal{A} := K[\mathcal{X}]$ the differential algebra generated by the coefficients of the fundamental matricial solution $\mathcal{X} := z^A$. The algebra $\mathcal{A}$ is generated by multivalued functions of the form z^α, where $\alpha \in \mathrm{Sp}(A)$, and maybe functions $z^\alpha \log$ if there are corresponding non trivial Jordan blocks. For any $(\gamma, \lambda) \in \hat{\pi}_1 = \mathrm{Hom}_{gr}(\mathbf{C}^*, \mathbf{C}^*) \times \mathbf{C}$, we know from chapter 12 that the map $\sigma_{\gamma,\lambda}$ sending z^α to $\gamma(e^{2i\pi\alpha})z^\alpha$ and $z^\alpha \log$ to $\gamma(e^{2i\pi\alpha})z^\alpha(\log + 2i\pi\lambda)$ is an element of $\mathrm{Aut}(\mathcal{A})$; and we also know that all elements of $\mathrm{Aut}(\mathcal{A})$ can be described in this way. Therefore, we obtain a surjective group morphism $(\gamma, \lambda) \mapsto \sigma_{\gamma,\lambda}$ from $\hat{\pi}_1$ to $\mathrm{Aut}(\mathcal{A})$. (The verification that it is indeed a group morphism is easy and left as an exercice to the reader.)

The matricial version of this morphism is obtained as follows. The fundamental monodromy matrix of $\mathcal{X} := z^A$ is $M := e^{2i\pi A}$. Let $M = M_sM_u = M_uM_s$ its Jordan decomposition. The map $\sigma \mapsto \mathcal{X}^{-1}\sigma(\mathcal{X})$ from $\mathrm{Aut}(\mathcal{A})$ to $\mathrm{GL}_n(\mathbf{C})$ is a group morphism with image the Galois group $\mathrm{Gal}(A)$. Composing it with the map $(\gamma, \lambda) \mapsto \sigma_{\gamma,\lambda}$ above, we get a representation:

$$\hat{\rho}_A : \begin{cases} (\gamma, \lambda) \mapsto \gamma(M_s)M_u^\lambda, \\ \hat{\pi}_1 \to \mathrm{GL}_n(\mathbf{C}), \end{cases} \qquad \text{where we put } \hat{\pi}_1 := \mathrm{Hom}_{gr}(\mathbf{C}^*, \mathbf{C}^*) \times \mathbf{C}.$$

Its image is the Galois group $\mathrm{Gal}(A) = \overline{\mathrm{Mon}(A)} = \overline{<M>}$. We now state without proof (and not even complete definitions !) the algebraic version of the Riemann-Hilbert correspondance.

Theorem 13.3.1 *The functor $A \rightsquigarrow \hat{\rho}_A$ induces a bijective correspondance:*

$$\left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{regular singular systems} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{isomorphism classes of} \\ \textit{rational representations of } \hat{\pi}_1 \end{array} \right\}$$

The Galois group of the system $X' = z^{-1}AX$ is Im $\hat{\rho}_A$.

□

Appendix A

Final test

One question on each chapter ! All documents are allowed.

Question on chapter 1: Compute the exponential of $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $a, b \in \mathbf{C}$.

Question on chapter 2: Find the unique power series such that $f(z) = (1-2z)f(2z)$ and $f(0) = 1$, and give its radius of convergence.

Question on chapter 3: Prove that $\dfrac{1}{e^z - 1}$ is meromorphic on $\mathbf{C}$ but that it is not the derivative of a meromorphic function.

Question on chapter 4: Let $A \in \mathrm{Mat}_n(\mathbf{C})$. Show that $\exp(A) = I_n$ if, and only if, A is diagonalisable and $\mathrm{Sp}(A) \subset 2\mathrm{i}\pi\mathbf{Z}$. (The proof that A is diagonalisable is not very easy.)

Question on chapter 5: Solve $z^2 f'' + zf' + f = 0$ on $\mathbf{C} \setminus \mathbf{R}_-$ and find its monodromy.

Question on chapter 6: Prove rigorously that $z^{\alpha}.z^{\beta} = z^{\alpha+\beta}$.

Question on chapter 7: What becomes the equation $zf'' + f' = 0$ at infinity ?

Question on chapter 8: Solve the equation $(1-z)\delta^2 f - \delta f - zf = 0$ by the method of Fuchs-Frobenius and compute its monodromy. (Just give the precise recursive formula for the Birkhoff matrix and a few terms.)

Question on chapter 11: Give a necessary and sufficient condition for the differential Galois group of equation $\delta^2 f + p\delta f + qf = 0$, where $p, q \in \mathbf{C}$, to contain unipotent matrices other than the identity matrix.

Question on chapter 12: Show that the Galois group of a regular singular system is trivial if, and only if, it admits a uniform fundamental matricial solution. Is the same condition valid for an irregular system ?

Appendix B

Answers to the final test

One question on each chapter ! All documents are allowed.

Question on chapter 1: Compute the exponential of $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $a, b \in \mathbf{C}$.

Answer to the question on chapter 1: The matrix is diagonalisable, with eigenvalues $a \pm b\mathrm{i}$:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = P \begin{pmatrix} a+b\mathrm{i} & 0 \\ 0 & a-b\mathrm{i} \end{pmatrix} P^{-1}, \text{ where } P = \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}.$$

Therefore:

$$\exp \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = P \begin{pmatrix} e^{a+b\mathrm{i}} & 0 \\ 0 & e^{a-b\mathrm{i}} \end{pmatrix} P^{-1} = \begin{pmatrix} e^a \cos b & -e^a \sin b \\ e^a \sin b & e^a \cos b \end{pmatrix}.$$

We put classically $\cos b := \dfrac{e^{\mathrm{i}b} + e^{-\mathrm{i}b}}{2}$ and $\cos b := \dfrac{e^{\mathrm{i}b} - e^{-\mathrm{i}b}}{2\mathrm{i}}$.

A direct computation is also possible, writing $A = aI_2 + bJ$ and noting that $J^2 = -I_2$. One finds $\exp(A) = e^a(cI_2 + sJ)$ where $c = 1 - \dfrac{b^2}{2!} + \dfrac{b^4}{4!} + \cdots = \cos b$ and $s = b - \dfrac{b^3}{3!} + \dfrac{b^5}{5!} + \cdots = \sin b$.

Question on chapter 2: Find the unique power series such that $f(z) = (1-2z)f(2z)$ and $f(0) = 1$, and give its radius of convergence.

Answer to the question on chapter 2: Writing $f = \sum_{n \geq 0} a_n z^n$, the conditions are equivalent to $a_0 = 1$ and $a_n = 2^n a_n - 2^n a_{n-1}$ for $n \geq 1$. The unique solution is:

$$a_n = \prod_{i=1}^{n} \frac{2^i}{2^i - 1} = \frac{1}{\prod_{i=1}^{n}(1 - 2^{-i})}.$$

Since $\frac{a_n}{a_{n-1}} = \frac{2^n}{2^n - 1} \underset{n \to +\infty}{\longrightarrow} 1$, the radius of convergence is 1.

Question on chapter 3: Prove that $\dfrac{1}{e^z-1}$ is meromorphic on $\mathbf{C}$ but that it is not the derivative of a meromorphic function.

Answer to the question on chapter 3: The denominator vanishes at all the $2k\mathrm{i}\pi$, $k \in \mathbf{Z}$ and only there, so the function is holomorphic on $\mathbf{C} \setminus 2\mathrm{i}\pi\mathbf{Z}$.
Since $\lim\limits_{z\to 2k\mathrm{i}\pi} \dfrac{z-2k\mathrm{i}\pi}{e^z-1} = \dfrac{1}{\exp'(2k\mathrm{i}\pi)} = 1$, the function has a simple pole at each $2k\mathrm{i}\pi$, therefore it is meromorphic on $\mathbf{C}$. (Other possible argument: $\mathbf{C}$ being connected, $\mathcal{M}(\mathbf{C})$ is a field.)
From the previous computation, the Laurent series expansion of the function at 0 is $\dfrac{1}{z}+$ terms with exponents ≥ 0. This is the derivative of no Laurent power series: indeed, the derivative of the term $a_n z^n$ is $n a_n z^{n-1}$ and can be z^{-1} for no values of n and a_n.

Question on chapter 4: Let $A \in \mathrm{Mat}_n(\mathbf{C})$. Show that $\exp(A) = I_n$ if, and only if, A is diagonalisable and $\mathrm{Sp}(A) \subset 2\mathrm{i}\pi\mathbf{Z}$. (The proof that A is diagonalisable is not very easy.)

Answer to the question on chapter 4: If A is diagonalisable and $\mathrm{Sp}(A) \subset 2\mathrm{i}\pi\mathbf{Z}$ then $\exp(A)$ is diagonalisable and $\mathrm{Sp}(\exp(A)) = \exp(\mathrm{Sp}(A)) \subset \{1\}$, so that $\exp(A) = I_n$.
If $\exp(A) = I_n$, then, since $\{1\} = \mathrm{Sp}(\exp(A)) = \exp(\mathrm{Sp}(A))$, one has $\mathrm{Sp}(A) \subset 2\mathrm{i}\pi\mathbf{Z}$. Write $A = A_s + A_n$ (Dunford decomposition) so that $\exp(A) = \exp(A_s)\exp(A_n)$ (multiplicative Dunford decomposition) so that $\exp(A_s) = \exp(A_n) = I_n$ (unicity). Write $A_n = PNP^{-1}$ where N is a strictly upper triangular matrix. Then $\exp(N) = P^{-1}\exp(A_n)P = I_n$. But, if $N \neq 0$ and if its first non trivial over diagonal is in position $j - i = d > 0$, the N^k for $k \geq 2$ have no non zero elements on that over diagonal. Therefore $N = 0$ and A is diagonlisable.

Question on chapter 5: Solve $z^2 f'' + z f' + f = 0$ on $\mathbf{C} \setminus \mathbf{R}_-$ and find its monodromy.

Answer to the question on chapter 5: Setting $X := \begin{pmatrix} f \\ zf' \end{pmatrix}$ and $A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the equation is equivalent to $X' = z^{-1}AX$. Therefore, its solutions on $\mathbf{C} \setminus \mathbf{R}_-$ are of the form $z^A X_0$ where $X_0 \in \mathbf{C}^2$ and z^A is defined with the principal determination of the logarithm. Here:

$$z^A = \exp\begin{pmatrix} 0 & \log z \\ -\log z & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} z^{\mathrm{i}} + z^{-\mathrm{i}} & \mathrm{i}(z^{\mathrm{i}} - z^{-\mathrm{i}}) \\ \mathrm{i}(z^{\mathrm{i}} - z^{-\mathrm{i}}) & z^{\mathrm{i}} + z^{-\mathrm{i}} \end{pmatrix}.$$

(This can be found with the help of the first exercice !) The conclusion is that the solutions of the equation are the linear combinations of z^{i} and $z^{-\mathrm{i}}$.
The monodromy matrix along the fundamental loop, expressed in this basis, is $\begin{pmatrix} e^{-2\pi} & 0 \\ 0 & e^{2\pi} \end{pmatrix}$.

Question on chapter 6: Prove rigorously that $z^{\alpha}.z^{\beta} = z^{\alpha+\beta}$.

Answer to the question on chapter 6: We consider $f := z^{\alpha}$ as the unique solution on $\mathbf{C} \setminus \mathbf{R}_-$ of $zf' = \alpha f$, $f(1) = 1$; similarly for $g := z^{\alpha}$ and $h := z^{\alpha+\beta}$. Then, $(fg)(1) = 1$ and, on $\mathbf{C} \setminus \mathbf{R}_-$:

$$z(fg)' = (zf')g + f(zg') = (\alpha f)g + f(\beta g) = (\alpha+\beta)(fg),$$

so $fg = h$ by unicity.

Question on chapter 7: What becomes the equation $zf'' + f' = 0$ at infinity ?

Answer to the question on chapter 7: We set $w := 1/z$ and $g(w) := f(z) = f(1/w)$. Then $g'(w) = -w^{-2} f'(1/w)$ and:

$$g''(w) = 2w^{-3} f'(1/w) + w^{-4} f''(1/w) = 2w^{-3} f'(1/w) + w^{-4}(-wf'(1/w)) = w^{-3} f'(1/w) = -w^{-1} g'(w),$$

so the equation becomes $wg'' + g' = 0$.

Question on chapter 8: Solve the equation $(1-z)\delta^2 f - \delta f - zf = 0$ by the method of Fuchs-Frobenius and compute its monodromy. (Just give the precise recursive formula for the Birkhoff matrix and a few terms.)

Answer to the question on chapter 8: We set $X := \begin{pmatrix} f \\ \delta f \end{pmatrix}$ and $A := \begin{pmatrix} 0 & 1 \\ \dfrac{z}{1-z} & \dfrac{1}{1-z} \end{pmatrix}$, so that the equation is equivalent to the system of the first kind $\delta X = AX$. We have $A(0) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, which has eigenvalues 0 and 1, so there is a resonancy. Taking $S := \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$, we find that:

$$B := S^{-1}AS - zS^{-1}S' = \begin{pmatrix} 0 & z \\ \dfrac{1}{1-z} & \dfrac{z}{1-z} \end{pmatrix}$$

meaning that $z^{-1}A = S[z^{-1}B]$. Now, $C := B(0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, so the new system has no resonancy. We are going to find a Birkhoff matrix $F = I_2 + zF_1 + \dots$ such that $z^{-1}B = F[z^{-1}C]$, *i.e.* $zF' = BF - FC$. To do that, we expand $B = B_0 + zB_1 + \dots$, where $B_0 = C$, $B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $B_k = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ for $k \geq 2$. Starting with $F_0 = I_2$, we are to solve recursively, for $k \geq 1$:

$$kF_k + F_kC - CF_k = B_1F_{k-1} + \dots + B_kF_0.$$

For instance, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} := F_1$ satisfies $F_1 + F_1C - CF_1 = B_1F_0 = B_1$, whence:

$$\begin{pmatrix} a+b & b \\ c+d-a & d-b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \Longrightarrow F_1 = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}.$$

Similarly, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} := F_2$ satisfies $2F_2 + F_2C - CF_2 = B_1F_1 + B_2F_0$, whence:

$$\begin{pmatrix} 2a+b & 2b \\ 2c+d-a & 2d-b \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix} \Longrightarrow F_2 = \begin{pmatrix} -3/2 & 1 \\ -5/2 & 3/2 \end{pmatrix}.$$

Once the Birkhoff matrix F is computed, one has $A/z = SF[C/z]$, so that a fundamental matricial solution of S_A is SFz^C. The monodromy relative to that solution is $e^{2i\pi C}$ (for the fundamental loop). Since $C^2 = 0$, this is equal to $I_2 + 2i\pi C = \begin{pmatrix} 1 & 0 \\ 2i\pi & 1 \end{pmatrix}$.

Question on chapter 11: Give a necessary and sufficient condition for the differential Galois group of equation $\delta^2 f + p\delta f + qf = 0$, where $p, q \in \mathbf{C}$, to contain unipotent matrices other than the identity matrix.

Answer to the question on chapter 11: The corresponding system is $\delta X = AX$ where $A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}$. It is a regular singular system in the most basic form, so a fundamental matricial solution is z^A. The characteristic polynomial of A is $T^2 + pT + q$. Its discriminant is $p^2 - 4q$.
If $p^2 - 4q \neq 0$, A has two distinct eigenvalues, so it is diagonalisable and $\mathrm{Gal}(A)$ contains only semi-simple elements.
If $p^2 - 4q = 0$, there is only one (double) eigenvalue; since A is not scalar (because of the 1 up right), it is not diagonalisable. Therefore, from the description given in the course, Gal contains unipotent elements.

Question on chapter 12: Show that the Galois group of a regular singular system is trivial if, and only if, it admits a uniform fundamental matricial solution. Is the same condition valid for an irregular system ?

Answer to the question on chapter 12: For a regular singular system, Gal is trivial if and only if Mon is: indeed, one implication is trivial since Mon $\subset$ Gal; and the other is a consequence of Schlesinger theorem. But we know that the triviality of the monodromy group is equivalent to having a fundamental matricial solution which is uniform.
For an irregular system, the condition remains necessary since Mon is included in Gal, but it is not a sufficient condition. For instance, the rank 1 system $f' = -z^{-2} f$ has a non trivial uniform solutions $e^{1/z}$, so its monodromy group is trivial; but its Galois group is $\mathbf{C}*$.

Appendix C

The original plan of the course

Plan of the course

First part: The Riemann-Hilbert correspondence

Complex analytic linear differential equations.

1. The complex logarithm (a complete study); the "characters" z^α.
2. Systems of rank n and equations of order n.
3. The theorem of Cauchy.
4. The sheaf of spaces of solutions.
5. The monodromy representation.

Local study of fuchsian systems.

1. Some useful tools (algebraic and transcendental).
2. Standard fuchsian systems; regular singular systems.
3. Local Riemann-Hilbert correspondence.

The global Riemann-Hilbert correspondence.

1. The "abelian" cases: rank one or one singularity in $\mathbf{C}$.
2. Riemann's theory of the hypergeometric equation.
3. The global Riemann-Hilbert correspondence.
4. Functorial view of the correspondence.

An algebraic view of monodromy.

1. Differential modules and their operations.
2. What can be seen algebraically of the monodromy group.
3. The density theorem of Schlesinger.

Second part: Affine algebraic groups

Linear representations of groups.

1. Reminders on groups and morphisms. The linear group.
2. Linear representations and their classification.
3. Can one recover a group from its representations ?

Basic affine algebraic geometry.

1. Algebraic subsets of $\mathbf{C}^n$. The Zariski topology.
2. Regular functions on algebraic sets. Algebras of functions.
3. How to recover an algebraic set from its algebra of functions.

Affine algebraic groups.

1. Affine algebraic groups and their algebras. Linear algebraic groups.
2. Rational representations of a linear algebraic group. The "little" theorem of Tannaka. The Jordan decomposition.
3. Reminders of multilinear algebra. The theorem of Chevalley.

Third part: Differential Galois Theory

The local Galois group for fuchsian equations.

1. The Galois group as an algebraic completion of the monodromy. Intepretation in terms of tannakian duality.
2. The universal Picard-Vessiot extension with meromorphic germs. The Galois group as an automorphism group.

Tannakian duality for differential modules.

1. The tensor category of fuchsian differential modules.
2. The tannakian universal Galois group. The tensor category of representations.

Picard-Vessiot Theory.

1. Universal Picard-Vessiot extensions.
2. The Galois correspondence.

Introduction to the irregular case: the Stokes phenomenon.

Appendix D

Standard notations

$\mathbf{N}, \mathbf{N}^*, \mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{R}_+, \mathbf{R}^*, \mathbf{R}_+^*, \mathbf{C}$

$\overset{\circ}{\mathrm{D}}(z_0, r)$

$n!$, $\binom{p}{n}$

Bibliography

[1] **Ahlfors L., 1979.** *Complex analysis*, McGraw-Hill.

[2] **Anosov D.V. and Bolibrukh A.A., 1994.** *The Riemann-Hilbert Problem*, *Aspects of Mathematics*, vol. E22, Vieweg.

[3] **Arnold V.I. and Ill'yashenko Yu.S., 1988.** *Ordinary differential equations*, in *Dynamical Systems I*, D.V. Anosov & V.I. Arnold (Eds.), Encyclopaedia of Mathematical Sciences, Vol. 1, Springer Verlag.

[4] **Beauville A., 1993.** Monodromie des systèmes différentiels linéaires à pôles simples sur la sphère de Riemann [d'après A. Bolibruch], Séminaire Bourbaki no 765, Astérisque 216, Société Mathématique de France.

[5] **Borel A., 1991.** *Linear Algebraic Groups*, Graduate Texts in Mathematics 126, Springer Verlag.

[6] **Bourbaki N., 1970.** *Algèbre, chapitres 1 à 3*, Diffusion C.C.L.S.

[7] **Deligne P., 1970.** *Equations différentielles à points singuliers réguliers*, Lecture notes in Mathematics 163, Springer Verlag.

[8] **Douady R. et Douady A., 2005.** *Algèbre et théories galoisiennes,* Nouvelle bibliothèque mathématique 4, Cassini.

[9] **Hilbert D., 1976.** *Mathematical developments arising from Hilbert problems*, Proceedings of symposia in pure mathematics, vol. XXVIII, part 1 and 2, American Mathematical Society.

[10] **Humphreys J.E., 1975.** *Linear Algebraic Groups*, Graduate Texts in Mathematics 21, Springer Verlag.

[11] **Iwasaki K., Kimura H., Shimomura S. et Yoshida M., 1990.** *From Gauss to Painlevé*, *Aspects of Mathematics*, vol. E16, Vieweg.

[12] **Kirillov, A., 1974** *Éléments de la théorie des représentations* Éditions Mir.

[13] **Lang S., 1984.** *Algebra (second edition)*, Addison-Wesley.

[14] **Mac Lane S., 1971.** *Categories for the Working Mathematician*, Graduate Texts in Mathematics no 5, Springer Verlag.

[15] **Mneimné R., 2006.** *Réduction des endomorphismes*, Calvage & Mounet.

[16] **Mneimné R. et Testard F., 1986.** *Introduction à la théorie des groupes de Lie classiques*, Hermann.

[17] **Mumford D., 1988.** *The red book of varieties and schemes*, Lecture Notes in Mathematics 1358, Springer Verlag.

[18] **van der Put M. and Singer M.F., 2003.** *Galois theory of linear differential equations*, Springer Verlag.

[19] **Ramis J.-P. et Warusfel A. (sous la direction de), 2009.** *Cours de Mathématiques pures et appliquées L3-M1, vol I*, De Boeck.

[20] **Riemann B., 1968.** *Oeuvres mathématiques*, Librairie Albert Blanchard.

[21] **Riemann B., 1851.** Principes fondamentaux pour une théorie générale des fonctions d'une grandeur variable complexe, *in* [20].

[22] **Riemann B., 1854.** Contribution à la théorie des fonctions représentables par la série de Gauss $F(\alpha,\beta,\gamma;x)$. *in* [20].

[23] **Sauloy J., 2005.** *Équations fonctionnelles analytiques dans le champ complexe*, Cours de DEA 2004-2005, URL `http://www.math.univ-toulouse.fr/~sauloy/`.

[24] **Serre J.-P., 1960.** *Groupes proalgébriques*, Publications Mathématiques de l'I.H.E.S no 7.

[25] **Serre J.-P., 1967.** *Représentations linéaires des groupes finis*, Collection Méthodes, Hermann.

[26] **Serre J.-P., 1970.** *Cours d'arithmétique*, Presses Universitaires de France.

[27] **Shafarevitch I.R., 1970** *Basic Algebraic Geometry*, Grundlehren der mathematischen Wissenschaften 213, Springer Verlag.

[28] **Springer T. A., 1991.** *Linear Algebraic Groups*, Progress in Mathematics 9, Birkhäuser.

[29] **Waterhouse W.C., 1979.** *Introduction to Affine Group Schemes*, Graduate Texts in Mathematics 66, Springer Verlag.

[30] **Whittaker E.T. and Watson G.N., 1927.** *A course of modern analysis*, Cambridge.

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