Galois Theory of q-Difference Equations: The "Analytical" Approach.

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Abstract

We show how a function theoretic approach to the Galois theory of fuchsian q-difference equations (in contrast to the purely algebraic approach due to van der Put and Singer) is possible and desirable. Based on classical complex geometry (theta functions and elliptic curves), it allows for a topological description of the Riemann-Hilbert-Birkhoff correspondance.

Keywords : q-difference equations - Differential Galois theory Tannakian Duality.

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To Marius van der Put, for his Doctorat honoris causa*,*
with best wishes for never ending research.

1 Introduction

Let q be a complex number such that $|q| > 1$. We fix $\tau \in \mathbf{C}$ such that $q = e^{-2\imath\pi\tau}$. For U a q-invariant subset of the Riemann sphere $\mathbf{S} = \mathbf{P}^1\mathbf{C}$ (that is, $qU = U$) and f a function on U, we write $\sigma_q f$ the function $z \mapsto f(qz)$. This notation can be extended to functions defined on a covering of U to handle the case of "multivalued functions"; it can also be extended to germs of meromorphic functions and to formal power series, thus defining automorphisms of the field $\mathbf{C}(\{z\})$ of convergent Laurent power series and of the field $\mathbf{C}((z))$ of formal Laurent power series.

We are interested in the classification of *rational linear q-difference systems* (or equations, which amounts to the same by the *cyclic vector lemma* of Birkhoff) over $\mathbf{S}$ under *rational equivalence*. Such a system has the form:

$$\sigma_q X = AX. \tag{1}$$

Here, the equation matrix A is invertible with rational coefficients: $A \in GL_n(\mathbf{C}(z))$. The seeked for solutions X are $n \times 1$ vector-valued functions, analytic over some q-invariant subset of $\mathbf{S}$, or maybe some covering of it if we want to allow for multivalued solutions. We also consider $n \times n$ matrix valued solutions ; such a solution, if invertible, we call a *fundamental solution.* Its column vectors make up, in old language, a *full complement of solutions.* The change of variable $X = FY$, $F \in GL_n(\mathbf{C}(z))$ gives rise to the system $\sigma_q Y = BY$, where $B = (\sigma_q F)^{-1} AF$: these systems are said to be *rationally equivalent.*

This text is intended to be a bird's eye view at a beautiful landscape originating in Birkhoff's work and 19th century complex geometry. A complete exposition and proofs can be found in [15]. I must apologize for omitting the two topics most relevant to the title of this workshop: confluence to differential equations (this is tackled in *loc. cit.* and also in [14]) and Stokes phenomena (this is tackled in [17] and in [13]). My excuse is that this work is in straight line with two talks at the previous workshop in Groningen, six years ago, that by Marius van der Put and that by Jean-Pierre Ramis.

Acknowledgements

Part of the present work constitutes the second part of my thesis, that was prepared under the direction of Jean-Pierre Ramis. But the whole investigation has been deeply influenced by his vision. I thank him heartily for his precious support. I have also been greatly helped by suggestions of the referees of that thesis, Yves André and Marius van der Put. It is therefore a very special pleasure to be invited to talk about this work at the occasion of the celebration of Marius' *Doctorat Honoris Causa.*

A first detailed exposition of this work was circulated in autumn 2000. It has since then been submitted for publication elsewhere. Then, in june 2001, as I came back from Groningen, Bernard Malgrange gave me a copy of the paper [2] of 1996, by Baranovsky and Ginzburg, emphasizing the use of vector bundles over an elliptic curve to classify q-difference equations. While our point of view is strikingly similar to theirs, there are the following differences: their study is purely local and formal, and involves no Galois theory; on the other hand, they study the classification with respect to more general gauge groups than $GL_n(\mathbf{C}(z))$.

1.1 Fundamental features

In what follows, the reader should try to keep in mind the analogy with the classical theory of complex ODE (ordinary differential equations) by Riemann, Fuchs and Frobenius.

1.1.1 Fields of solutions and fields of constants

We shall consistently solve (1) with uniform functions, that is, functions meromorphic over $\mathbf{C}^*$, which is, of course, generally impossible for complex ODE! Accordingly, two fundamental solutions X and Y of (1) are connected by a matrix $P = Y^{-1}X$ with coefficients in the field $\mathcal{M}(\mathbf{C}^*)^{\sigma_q}$ of q-invariant meromorphic functions. The change of variables $z = e^{2\imath\pi x}$ allows one to identify this field with the field of elliptic functions $\mathcal{M}(\mathbf{E})$, where $\mathbf{E}$ is the complex torus (or elliptic curve) $\mathbf{C}/\left(\mathbf{Z} + \mathbf{Z}\tau\right) = \mathbf{C}^*/q^{\mathbf{Z}}$.

1.1.2 Elementary equations

As in classical theory, one must first solve one-dimensional and unipotent two-dimensional equations.

(i) The basic brick is Jacobi's multiplicative theta function:

$$\Theta_q(z) = \sum_{n \in \mathbf{Z}} q^{-\frac{n(n-1)}{2}} z^n = (q^{-1}; q^{-1})_\infty (z; q^{-1})_\infty (q^{-1} z^{-1}; q^{-1})_\infty,$$

where one puts $(x; q^{-1})_\infty = \prod_{n \geq 0} (1 - x q^{-n})$ [1]. It satisfies the following q-difference equation: $\sigma_q \Theta_q = -qz\Theta_q$.

(ii) The q-character $e_{q,c}(z) = \frac{\Theta_q(z)}{\Theta_q(c^{-1}z)}$ is solution of the equation $\sigma_q e_{q,c} = c e_{q,c}$.

(iii) The q-logarithm $l_q(z) = z\frac{\Theta_q'(z)}{\Theta_q(z)}$ is solution of the equation $\sigma_q l_q = l_q + 1$.

Note that the mere presence of all the $\frac{e_{q,c}e_{q,d}}{e_{q,cd}}$ is enough to generate the whole field $\mathcal{M}(\mathbf{E})$. We shall comment on this further below (in 2.1).

1.1.3 Local classification

Along with rational equivalence, one may also allow for gauge transformations $F \in GL_n(\mathbf{C}(\{z\}))$, that is with coefficients meromorphic near 0 ; the functional equation $\sigma_q F = AFB^{-1}$ then shows the coefficients to be actually meromorphic on the whole of $\mathbf{C}$: $F \in GL_n(\mathcal{M}(\mathbf{C}))$. This defines *meromorphic equivalence* at 0. A similar notion is defined at ∞, involving $w = \frac{1}{z}$.

1.1.4 Fuchsian equations

Call *strictly fuchsian* at 0 (resp. at ∞) a q-difference system A such that $A(0) \in GL_n(\mathbf{C})$ (resp. $A(\infty) \in GL_n(\mathbf{C})$) and *fuchsian* at 0 (resp. at ∞) one that is meromorphically equivalent to a strictly fuchsian system at 0 (resp. at ∞). Then a system that is fuchsian both at 0 and ∞ (for short: fuchsian, or fuchsian over $\mathbf{S}$) can be proved to be rationally equivalent to a system that is strictly fuchsian both at 0 and ∞ (for short: strictly fuchsian). Fuchsianity is essentially equivalent to the existence of fundamental solutions with moderate growth at 0 and ∞, as opposed to theta-like growth, like $z^{\log z}$ (*see* [10]), but such growth conditions are not simple to state precisely. Fuchsianity can also be tested with the help of a *Newton polygon* (*see* [14], [16]).

1.2 Birkhoff's classification scheme

In analogy with Riemann's classification scheme for complex ordinary differential equations, by local data at the singularities (for local classification) and monodromy transformations arising from the analytic continuation of local solutions, the so-called "connection formulae" (for global classification), Birkhoff defined in [3] the "generalized Riemann

[1]The equality of these two expressions, by no means trivial, is Jacobi's celebrated *triple product formula*.

problem" for q-difference equations. He solved it for *fuchsian equations* under innocuous (generically true) assumptions. He did it in a unified manner for ordinary differential, difference and q-difference equations. We now describe Birkhoff's classification scheme, as slightly revised by us in [14]. The idea of Birkhoff is to use local solutions at the two only q-invariant points of $\mathbf{S}$, 0 and ∞, and to classify fuchsian systems using linear data at 0 and ∞ and *one* matrix connecting a local solution at 0 to one at ∞. Here are the main steps.

1. Any strictly fuchsian system is rationally equivalent to one that is *non resonant* at 0. This means that two *exponents*, that is, two eigenvalues of $A(0) \in GL_n(\mathbf{C})$, cannot be q-congruent if they are distinct. Then such a non resonant system is Birkhoff equivalent (that is, with a gauge transformation $F \in GL_n(\mathbf{C}\{z\})$ such that $F(0) = I_n$) to one with constant coefficients. Composing the two gauge transformations, we see that any fuchsian system is meromorphically equivalent to a constant coefficient system.

2. Any constant coefficient system can be reduced by linear algebra (Jordan reduction) to 1-dimensional systems: $\sigma_q f = cf$ (where c is an exponent) and, if its matrix is not semi-simple, 2-dimensional unipotent systems: $\sigma_q g = g + 1$. From this, using q-characters and q-logarithms, we get a canonical fundamental solution $e_{q,A}$ for each constant coefficient system $A \in GL_n(\mathbf{C})$. This canonical solution should be compared with the z^A of classical theory.

3. Each fuchsian system is therefore endowed with a local solution at 0: $X^{(0)} = M^{(0)} e_{q,A^{(0)}}$, where $M^{(0)} \in GL_n(\mathcal{M}(\mathbf{C}))$ and $A^{(0)} \in GL_n(\mathbf{C})$, and a local solution at ∞: $X^{(\infty)} = M^{(\infty)} e_{q,A^{(\infty)}}$, where $M^{(\infty)} \in GL_n(\mathcal{M}(\mathbf{C}_\infty))$ and $A^{(\infty)} \in GL_n(\mathbf{C})$. The Jordan structures of $A^{(0)}$ and $A^{(\infty)}$ are the the required "local linear data". We note that $X^{(0)}, X^{(\infty)} \in GL_n(\mathcal{M}(\mathbf{C}^*))$.

4. The *connection matrix of Birkhoff* is then defined to be:
$$P = \left(X^{(\infty)}\right)^{-1} X^{(0)}.$$
It has coefficients in the elliptic function field $\mathcal{M}(\mathbf{E})$.

Now, Birkhoff's theorem says that, given the linear data at 0 and at ∞ and an invertible elliptic matrix, one can recover a system (1) well defined up to rational equivalence. A more precise formulation can be found in [14]. A very characteristic feature of our version of Birkhoff's theorem is the following: when one seeks to make explicit the non unicity of the triple attached to a particular equation, one finds that *the set of those triples is acted upon by a subgroup of $GL_n(\mathbf{C})$ and that this action is transitive.* This is due to the fact that, e.g., $e_{q,c}$ is a "legal" solution to the character equation, while $\frac{e_{q,cd}}{e_{q,d}}$ is not. The benefit we get is the hope for a Galois group defined over $\mathbf{C}$ (vs the field of constants $\mathcal{M}(\mathbf{E})$). The price to pay is that *solutions cannot be multiplied.*

2 The Galois way with or without function theory

Birkhoff's method seems to lend itself easily to a reformulation in the style of Riemann-Hilbert's classification scheme, via local systems and representations of the fundamental group. The local linear data should allow one to compute local monodromy transformations around 0 and ∞, so that a *groupoid with two base points* naturally appears. Then these base points should be connected by paths from 0 to ∞: here, the representation has a natural image, the connection matrix. But the latter has elliptic coefficients, so that one does not end up with algebraic matrices over the complex numbers, as in the classical case. The field of "constants" of the theory, $\mathcal{M}(\mathbf{C}^*)^{\sigma_q} = \mathcal{M}(\mathbf{E})$, is too big. This seems unescapable.

2.1 Picard-Vessiot theory

The first break through the difficulties caused by the big constant field was made by Etingof in [6]. He succesfully adapted Picard-Vessiot theory to the case of q-difference equations, but with a rather restrictive assumption: he considered only *regular* systems, those such that $A(0) = A(\infty) = I_n$. This condition is essentially equivalent to the existence of local solutions $X^{(0)}, X^{(\infty)}$ respectively meromorphic at $0, \infty$. Etingof defines and builds Picard-Vessiot extensions and shows the related q-difference Galois group to be generated by the *values* (at authorized points) $P(a)^{-1}P(b) \in GL_n(\mathbf{C})$, where P is the (elliptic) connection matrix. This shows a way to produce complex matrices from elliptic ones.

However, in the case of non regular systems, elliptic functions spontaneously arise in yet another way: if $c, d \in \mathbf{C}^*$, then, along with the "legal" solution $e_{q,cd}$ to the equation $\sigma_q f = cdf$, there is also $e_{q,c}e_{q,d}$, so that $\phi(c,d) = \frac{e_{q,c}e_{q,d}}{e_{q,cd}}$ is elliptic. Any field containing these basic solutions will contain *all* elliptic functions, and it is easy to prove that there is no way to trivialize the cocycle $\phi(c,d)$ while using "true functions", so that there seems to be no hope for a Picard-Vessiot theory with a group defined over $\mathbf{C}$ even in the fuchsian case.

The second breakthrough was accomplished by van der Put and Singer in [8], using *symbolic solutions*. Taking symbols e_c required to satisfy $\sigma_q e_c = ce_c$ and, at the same time, $e_c e_d = e_{cd}$ allows them to build a Picard-Vessiot theory with constant field $\mathbf{C}$. The minor drawback is that they have to use non integral algebras instead of fields. But they are able to solve the problem in total generality, including the case of *irregular* (non fuchsian) equations. They also give a tannakian description of the q-difference Galois group and show how it is related to the values of the connection matrix. Thus, from an algebraic point of view, the problem could be considered as settled.

2.2 On the usefulness of retaining true functions

There are, however, quite a few good reasons to wish for a solution in classical style, using analytical functions. Here are some.

1. The oldest historical motivation for the q-world lies in magic identities by Gauss, Euler, Jacobi, Ramanujan ... (*see* [9] and the "bible" [7]). These involve classical analytical functions. It would be fascinating to understand how Ramanujan found some of these identities. This might, at the moment, look like a dream - but who knows ...

2. Another motivation is the link of q-difference equations to ordinary differential equations through q-analogies. When $q \to 1$, q-analogs of special functions (like Heine's *basic hypergeometric series*, *see* [7]) "tend to" their classical counterpart. One may wish to follow Galois groups along such a *confluency*. This is of obvious interest in mathematical physics, with the present ubiquity of q-deformations. This point of view has been, I believe, quite justified by our first results presented in [14] as well as the additional results given in [15]. Recent work by Changgui Zhang (*see* [20]) on the q-Bessel function bears promises for the irregular case.

3. q-difference equations are a possible intermediate (and easier) step to understand the mysterious analogy between irregular linear ordinary differential equations and wildly ramified phenomena in positive characteristic algebraic geometry (*see* [12]). This strategy seems not to be totally out of reach, this particular line of thought taking place within the investigations of Jean-Pierre Ramis (who suggested the present work).

4. It is natural to uniformize algebraic objects with analytic functions and to look for explicit solutions. This takes us into the realm of classical complex function theory, where algebraic geometry took birth, with its wealth of beautiful geometric objects and algebraic structures. This is largely illustrated in this work, with the appearance (and central importance) of Jacobi's theta functions, elliptic functions and complex elliptic curves in the landscape. Moreover, the possibility of a geometric interpretation of the galoisian automorphisms that we explicitly build allows us to exhibit a reasonable candidate for the role of *fundamental group*, that is a finitely generated and finitely presented Zariski-dense subgroup of the Galois group, thereby giving an analogy with Schlesinger's density theorem and with the topological flavour of the classical Riemann-Hilbert correspondance, where fuchsian differential equations are classified by monodromy representations.

As noted before, the use of "true functions" as q-characters forces on us a big constant field, in our version of the theory, the whole field of elliptic functions $\mathcal{M}(\mathbf{E})$. Thus, a Picard-Vessiot theory seems to be out of reach in this framework. Yet, our version of the classification theorem of Birkhoff indicates very clearly the possibility of a group defined over $\mathbf{C}$. Indeed, as we said, automorphisms of fuchsian objects are classified by complex matrices. Therefore, to produce a Galois group, we turn to Tannaka duality (*see* [5], [4]). This has already been succesfully used in similar context twice: by van der Put and Singer in [8] for one, then by Yves André in his work [1], where semicontinuity results are proved for difference and differential Galois groups. Our goal is therefore to give a tannakian formulation of the classification theorem of Birkhoff, while using as basic objects uniform analytic functions.

3 Galois theory with function theory, first attack

We shall build a Galois theory, while keeping in mind the *classical model* for the relationship between the topology (representations of the monodromy group) and the algebraic geometry (regular representations of the Galois group), as summarized now.

The classical density theorem. - We describe it locally. We consider fuchsian differential systems:

$$z\frac{d\,X}{d\,z} = A\,X\ , \quad A(0) \in M_n(\mathbf{C}) \tag{2}$$

locally at 0. The classification of such systems boils down to the classification of their monodromy representations, that is, representations of the fundamental group of the germ $(\mathbf{C}^*, 0)$. Thus, the category of fuchsian differential systems, along with its natural abelian and tensor structures is equivalent to the category $\mathcal{R} = Rep_{\mathbf{C}}(\mathbf{Z})$ of finite dimensional complex representations of $\mathbf{Z}$: this is just Riemann-Hilbert correspondance. In this equivalence, the "fibre functor":

system (2) $\rightsquigarrow$ space of solutions

corresponds to the forgetful functor:

complex representation of $\mathbf{Z} \rightsquigarrow$ underlying vector space.

The monodromy action of $\mathbf{Z}$ on the space of solutions of (2) corresponds to the tautological action of $\mathbf{Z}$ by automorphisms of the forgetful functor. These automorphisms are easily seen to be tensor compatible.

Now, the category $\mathcal{R}$ is known to be the same as the category of regular representations of a uniquely defined complex algebraic group, the *proalgebraic hull* of $\mathbf{Z}$:

$$\mathbf{Z}^{alg} = Hom_{gr}(\mathbf{C}^*, \mathbf{C}^*) \times \mathbf{C}.$$

Elements of $\mathbf{Z}^{alg}$ should be considered as automorphisms of the fibre functor "space of solutions" Here, an element γ of the semi-simple component $Hom_{gr}(\mathbf{C}^*, \mathbf{C}^*)$ acts on the space of solutions of (2) through the matrix $\gamma(e^{2\imath\pi A_s(0)})$, where $A_s(0))$ is the semi-simple part of $A(0)$ (the notation $\gamma(S)$ for a semi-simple invertible matrix S means that γ has been applied to the eigenvalues of S). An element λ of the unipotent component $\mathbf{C}$ acts by $A_u(0)^\lambda$, where $A_u(0)$ is the unipotent part of $A(0)$.

The monodromy group $\mathbf{Z}$ embeds in its proalgebraic hull by $n \mapsto (z \mapsto z^n, n)$. Thus, the fundamental loop $1 \in \mathbf{Z}$ can be thought as composed of the unipotent loop:

$$1 \in \mathbf{Z}_u^{alg} = \mathbf{C}$$

and of the semi-simple loop:

$$Id_{\mathbf{C}^*} \in \mathbf{Z}_s^{alg} = Hom_{gr}(\mathbf{C}^*, \mathbf{C}^*).$$

It is then easy to recover the monodromy action on solutions, looking at the basic semi-simple equation $zx' = cx$ and at the basic unipotent equation: $z(zx')' = 0$. Picard-Vessiot theory actually computes $\mathbf{Z}^{alg}$ and Schlesinger's density theorem says that $\mathbf{Z}$ is Zariski-dense in $\mathbf{Z}^{alg}$, thus giving this Galois group its true topological meaning [2].

3.1 Tannakian constructions

In the first approach, we follow Birkhoff's scheme litterally. The category $\mathcal{E}_f$ of fuchsian q-difference equations is endowed with its natural tensor structure and becomes a rigid tannakian tensor category, with at least two obvious fibre functors:

[2] The density theorem for *irregular* equations is the content of Ramis' theory of the wild differential group.

1. The forgetful functor : it forgets the equation and keeps the underlying $\mathbf{C}(z)$-vector space. But it can only help to define a Galois group over $\mathbf{C}(z)$.
2. The functor of solutions in $\mathcal{M}(\mathbf{C}^*)$. But this gives $\mathcal{M}(\mathbf{E})$-vector spaces and can only produce a Galois group defined over $\mathcal{M}(\mathbf{E})$.

Following closely Birkhoff, we encode a system (1) by a triple $(A^{(0)}, P, A^{(\infty)})$ made up of two constant matrices meromorphically equivalent to A (respectively at 0 and at ∞) and the connection matrix P. Actually, the choice of the triple is by no means canonical, but it is easy to overcome this using an intermediate category of solutions.

3.1.1 Morphisms

Using our version of Birkhoff's classification theorem, one can define morphisms between such triples as pairs $(R^{(0)}, R^{(\infty)})$ of *complex* matrices, in such a way that the resulting category $\mathcal{C}$ is equivalent to $\mathcal{E}_f$ and that the natural forgetful functors ω_0 and ω_∞ on $\mathcal{C}$ are exact and faithful. This is the "benefit" alluded at the end of 1.2. The "price to pay" will come immediately:

3.1.2 Tensor structure

If we want to have a *tensor* equivalence of categories, we must choose a *twisted* tensor product of solutions, hence of connection matrices. Indeed, the tensor product of the character equations $\sigma_q f = cf$ and $\sigma_q f = df$ is $\sigma_q f = cdf$. It has as "legal" solution, not the tensor product $e_{q,c} \otimes e_{q,d} = e_{q,c} e_{q,d}$ of the legal solutions, but, instead, $e_{q,cd}$, which is a twisted form:

$$\begin{aligned} e_{q,c} \underline{\otimes} e_{q,d} &\underset{def}{=} e_{q,cd} \\ &= (e_{q,c} \otimes e_{q,d}) \, (\phi(c,d))^{-1} \end{aligned}$$

More generally, legal solutions at 0 (resp. at ∞) of the systems of equations with constant coefficients A and B inherit a tensor product twisted by a cocycle $\Phi(A_s^{(0)}, B_s^{(0)})$ (resp. $\Phi(A_s^{(\infty)}, B_s^{(\infty)})$). Accordingly the tensor product of connection matrices has to be twisted too; with natural notations:

$$P \underline{\otimes} Q = \Phi(A_s^{(\infty)}, B_s^{(\infty)}) \, (P \otimes Q) \left(\Phi(A_s^{(0)}, B_s^{(0)}) \right)^{-1}.$$

We can now check that ω_0 and ω_∞ are tensor compatible and we are rewarded by the

Theorem. - *The tensor categories $\mathcal{E}_f$ and $\mathcal{C}$ are equivalent. The category $\mathcal{C}$ is tannakian over $\mathbf{C}$, with fibre functors ω_0 and ω_∞.*

As a consequence, we should have a Galois *groupoid* with two base points, 0 and ∞:

$$\begin{tikzcd} Aut^{\otimes}(\omega^{(0)}) & \omega^{(0)} \arrow[l, phantom, "\circlearrowleft"] \arrow[r, shift left, "Iso^{\otimes}(\omega^{(0)}{,}\omega^{(\infty)})"] & \omega^{(\infty)} \arrow[l, shift left, "Iso^{\otimes}(\omega^{(\infty)}{,}\omega^{(0)})"] \arrow[r, phantom, "\circlearrowright"] & Aut^{\otimes}(\omega^{(\infty)}) \end{tikzcd}$$

Here, we mean by *groupoid* with base space X the following data:

- for each pair $(x, y) \in X^2$, a set $A(x, y)$;
- for each point $x \in X$, a special element $I_x \in A(x, x)$;
- for each triple $(x, y, z) \in X^3$, a composition law $A(x, y) \times A(y, z) \to A(x, z)$.

These data are subject to the following requirements:

1. the composition law is associative;
2. each I_x is a left and right identity element;
3. each $f \in A(x, y)$ is invertible.

This amounts to ask for a small category all of whose arrows are isomorphisms.

There is a corresponding notion in algebraic geometry, *see* [4]: the base space and all of the arrow sets $A(x, y)$ are collectively endowed with a structure of algebraic variety and all relevant maps are morphisms. In our case, this structure is rather obvious in the classical sense: the base set $X = \{0, \infty\}$ is discrete and each $A(x, y) = Iso^{\otimes}(\omega^{(x)}, \omega^{(y)})$ can be identified with an affine variety of matrices.

3.2 The Galois groupoid

We shall now start filling our groupoid with as many galoisian elements as possible.

3.2.1 Local components

We find, within $Aut^{\otimes}(\omega^{(0)})$, a part looking very much like $\mathbf{Z}^{alg}$. We give it here dogmatically, it will be justified more naturally in the second attack (in 4.1):

$$G^{(0)} = Hom_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*) \times \mathbf{C}.$$

Its unipotent component $G_u^{(0)}$ is the same as that of $\mathbf{Z}^{alg}$, but its semi-simple component $G_s^{(0)}$ is made up of those elements of $\mathbf{Z}_s^{alg}$ that send q to 1. This is a first hint that the underlying space is the germ $(\mathbf{C}^*, 0)$ quotiented by $q^{\mathbf{Z}}$, an infinitesimal elliptic curve (see 3.3.1). There is a similar local group at ∞, $G^{(\infty)}$: it is isomorphic to $G^{(0)}$.

3.2.2 Connection component

If we try to follow Etingof's path, we should consider, for all authorized a, the mapping:

$$(A^{(0)}, P, A^{(\infty)}) \rightsquigarrow P(a).$$

It is easily seen to be a functorial isomorphism of ω_0 with ω_∞ (at least, their restrictions to the objects of $\mathcal{C}$ for which a is not a singular point). However, due to the twisting of the tensor product, it is not tensor preserving. We have to twist the connection matrix to get an element of $Iso^{\otimes}(\omega^{(0)}, \omega^{(\infty)})$, using instead:

$$(A^{(0)}, P, A^{(\infty)}) \rightsquigarrow \check{P}(a) = \left(\Psi_a(A_s^{(\infty)})\right)^{-1} P(a) \left(\Psi_a(A_s^{(0)})\right).$$

Here, Ψ_a is the extension to semi-simple invertible matrices of a function ψ_a acting on their eigenvalues; the function ψ_a is chosen so that $g_a \;:\; c \mapsto \frac{e_c(a)}{\psi_a(c)}$ is a morphism of groups from $\mathbf{C}^*$ to itself sending q to a. This does provide us with galoisian isomorphisms.

Density lemma. - *The galoisian elements found so far generate a Zariski-dense subgroupoid of the Galois groupoid.*

Note that different choices for g_a merely change our *connection* galoisian isomorphisms by *local* galoisian automorphisms as found above. Also note that in the case of *regular* equations, ψ_a acts trivially (it is to be applied to the exponents c, all of which are equal to 1) and $\breve{P} = P$, the local component being moreover trivial: we recover Etingof's results.

3.3 The monodromy groupoid

The Zariski-dense subgroupoid we just constructed is by no means a satisfactory answer, since it is so far from a finitely presented discrete groupoid as any reasonable monodromy groupoid should be. Therefore, we now try to fix that.

3.3.1 Local components

We can define particular elements of the local component $G^{(0)}$ (the case of ∞ is entirely similar). As for the unipotent part, $1 \in \mathbf{C}$ will do, since it clearly generates a Zariski-dense subgroup. As for the semi-simple component, we define two particular "semi-simple loops at 0". We can split any non zero complex number in a unique way as uq^y , with $|u| = 1$ and $y \in \mathbf{R}$; here, of course, we define q^y as $e^{-2\imath\pi\tau y}$. We then put:

$$\begin{cases} \gamma_1(uq^y) = u \\ \gamma_2(uq^y) = e^{2\imath\pi y} \end{cases}$$

It is easy to check that we do get elements of $G_s^{(0)}$.

Local density theorem. - *The loops γ_1 and γ_2 generate a Zariski-dense subgroup of $G_s^{(0)}$.*

These loops have a geometrical meaning related to the "infinitesimal elliptic curve" and were anticipated long ago by J.-P. Ramis in [11]: γ_2 is tied to the non unicity of choice of the logarithm, it represents the plain classical loop around 0 in $\mathbf{C}^*$; γ_1 is tied to the defect of ellipticity of the twisted connection matrix, it represents the beginning of the movement from 0 to ∞ along a q-spiral.

3.3.2 Connection component

Here, the matter is very subtle and very difficult. We do get a precise description in the restricted case of regular abelian equations, using geometric class field theory as exponded by Serre in [18]. But even this case is rather complicated and we refer to [15].

4 Galois theory with function theory, second attack

Here, we try to get rid of the difficulties encountered while multiplying solutions. We shall only take care of the Galois group, not of the monodromy.

4.1 Local theory

If we ask what, in the character $e_{q,c}$ is intrinsic of the exponent c, we see that it is its *congruence class modulo invertible elliptic functions*, that is, a *divisor class over the elliptic curve*. It is then natural to guess that what is intrinsic, in the canonical solution $e_{q,A}$, of the system A, is a vector bundle.

We realize that guess as follows. We start from a system (1) with matrix A. It is meromorphically equivalent to a system with constant matrix $A^{(0)}$. To give this a categorical meaning, we first *thicken* the category $\mathcal{E}_f$ into a category $\mathcal{E}_f^{(0)}$ which has the same objects as $\mathcal{E}_f$ but where we authorize all morphisms meromorphic at 0 (instead of rational). This can be seen as some kind of *localisation at* 0.

Each object in $\mathcal{E}_f^{(0)}$ is then isomorphic to a *flat* object, that is, one with constant coefficients. We shall see later the reason for calling these flat. Therefore, the category $\mathcal{E}_f^{(0)}$ has as an essential subcategory the category $\mathcal{P}^{(0)}$ of its flat objects. We now proceed to study the tannakian category $\mathcal{P}^{(0)}$.

Starting from an object $A \in GL_n(\mathbf{C})$ of $\mathcal{P}^{(0)}$, we introduce the equivalence relation $\sim_A$ on $\mathbf{C}^* \times \mathbf{C}^n$ generated by the relations:

$$\forall (z, X) \in \mathbf{C}^* \times \mathbf{C}^n \ , \ (z, X) \sim_A (qz, AX).$$

The first projection $\mathbf{C}^* \times \mathbf{C}^n \to \mathbf{C}^*$ is compatible with the action of $q^{\mathbf{Z}}$ on $\mathbf{C}^*$ and, quotienting, we define a holomorphic vector bundle of rank n:

$$F_A \underset{def}{=} \frac{\mathbf{C}^* \times \mathbf{C}^n}{\sim_A} \to \mathbf{E} = \frac{\mathbf{C}^*}{q^{\mathbf{Z}}}.$$

This is a flat vector bundle over $\mathbf{E}$. We take here *flat* to mean a bundle that can be described by transition matrices with constant coefficients: this is known to be equivalent to a differential geometric condition (existence of an integrable connection; the name comes from that setting) and, in the case of holomorphic bundles over algebraic curves, to a topological condition (all indecomposable factors have zero degree).

Lemma. - *We thus obtain an equivalence of $\mathcal{E}_f^{(0)}$ with the category $Fib_p(\mathbf{E})$ of flat vector bundles over $\mathbf{E}$ (but, as morphisms, all holomorphic morphisms).*

On the other hand, note that our category $\mathcal{P}^{(0)}$ looks very much like the category $\mathcal{R}$ of complex representations of $\mathbf{Z}$, it only has more morphisms: indeed, all *equivariant* matrix maps F (such that $F(qz)B = AF(z)$) are allowed, while morphisms in $\mathcal{R}$ have to be *constant* matrices This is actually a particular case of Weil's correspondance between complex representations of the fundamental group of a Riemann surface $\mathbf{X}$ and flat vector bundles on $\mathbf{X}$, *see* [19]. Note that Weil does make the distinction between the two kinds of isomorphisms of representations.

This means that the Galois group of $\mathcal{E}_f^{(0)}$ is the subgroup of the Galois group $\mathbf{Z}^{alg}$ of $\mathcal{R}$ made up of those elements that are functorial with all equivariant morphisms, not only constant ones.

Theorem. - *The Galois group of $\mathcal{E}_f^{(0)}$ is the group $G^{(0)}$ previously introduced.*

The condition $q \mapsto 1$ imposed on the elements of $G_s^{(0)}$ is easily found by taking a semi-simple element of $\mathbf{Z}^{alg}$ and requiring that it be functorial with respect to the following morphism in $\mathcal{P}^{(0)}$:

$$1 \xrightarrow{z} q.$$

One can then prove that the condition $q \mapsto 1$ is in turn sufficient to ensure functoriality with respect to all equivariant morphisms. Actually, the category $\mathcal{P}^{(0)}$ has plenty of fibre functors: as many as points $a \in \mathbf{C}^*$. The fibre functor $\omega_a^{(0)}$ thus sends an object $A^{(0)}$ to its underlying $\mathbf{C}^n$ and a morphism $F : A^{(0)} \to B^{(0)}$ to the matrix $F(a)$. This allows one to define a *local Galois groupoid* with base set $\mathbf{C}^*$. The condition for the semi-simple parts of the galoisian isomorphisms from $\omega_a^{(0)}$ to $\omega_b^{(0)}$ is that they send q to $\frac{b}{a}$.

4.2 Global theory

Since we have localized $\mathcal{E}_f$ at 0 and likewise at ∞, we expect to get its Galois group by glueing $G^{(0)}$ and $G^{(\infty)}$: this is a kind of Van Kampen's theorem.

Lemma. - *The category $\mathcal{E}_f$ is equivalent to the category of triples $(A^{(0)}, F, A^{(\infty)})$ where $A^{(0)}, A^{(\infty)}$ respectively are objects of $\mathcal{E}_f^{(0)}$, $\mathcal{E}_f^{(\infty)}$ and F is a meromorphic bijective map of the corresponding flat vector bundles. The morphisms are pairs of morphisms in $\mathcal{E}_f^{(0)}$ and $\mathcal{E}_f^{(\infty)}$ with natural commutativity conditions.*

Now, here, the tensor structure is *the natural one.* As for the connecting component, we find that the following mapping:

$$(A^{(0)}, F, A^{(\infty)}) \rightsquigarrow F(a)$$

is a tensor isomorphism from $\omega_a^{(0)}$ to $\omega_a^{(\infty)}$ and we have a density lemma similar to the one encountered above. All twisting is gone !

The following picture is an attempt to relate the results of our two attacks. It shows the diagram of commutativities within the two new local groupoids and with respect to the old fibre functors, when evaluated on an object $\mathcal{X} = (A^{(0)}, F, A^{(\infty)})$ at the three points $a, b, c \in \mathbf{C}^*$:

$$
\begin{tikzcd}
 & & \omega_a^{(0)}(\mathcal{X}) \arrow[rr, "F(a)"] \arrow[dd, "g_{\frac{b}{a}}(\overline{A_s^{(0)}})"] & & \omega_a^{(\infty)}(\mathcal{X}) \arrow[dr, "g_a(\overline{A_s^{(\infty)}})"] \arrow[dd, "g_{\frac{b}{a}}(\overline{A_s^{(\infty)}})"] & \\
\omega^{(0)}(\mathcal{X}) \arrow[urr, "g_a(\overline{A_s^{(0)}})"] \arrow[r, "g_c(\overline{A_s^{(0)}})"] \arrow[drr, "g_b(\overline{A_s^{(0)}})"'] & \omega_c^{(0)}(\mathcal{X}) \arrow[ur, "g_{\frac{a}{c}}(\overline{A_s^{(0)}})"] \arrow[rr, "F(c)"] & & \omega_c^{(\infty)}(\mathcal{X}) \arrow[ur, "g_{\frac{a}{c}}(\overline{A_s^{(\infty)}})"] \arrow[rr, "g_c(\overline{A_s^{(\infty)}})"] & & \omega^{(\infty)}(\mathcal{X}) \\
 & & \omega_b^{(0)}(\mathcal{X}) \arrow[ul, "g_{\frac{c}{b}}(\overline{A_s^{(0)}})"] \arrow[rr, "F(b)"] & & \omega_b^{(\infty)}(\mathcal{X}) \arrow[ul, "g_{\frac{c}{b}}(\overline{A_s^{(\infty)}})"] \arrow[ur, "g_b(\overline{A_s^{(\infty)}})"'] &
\end{tikzcd}
$$

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