Classification of q-difference equations using holomorphic vector bundles

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Abstract

Let q a complex number such that $0 < |q| < 1$. With every analytic q-difference module is associated a holomorphic vector bundle over the corresponding elliptic curve $\mathbf{E}_q := \mathbf{C}^*/q^Z$.
For fuchsian q-difference modules, this allows for analytic classification, including the calculation of the Galois group (Baranovsky-Ginzburg-Kontsevich).
For arbitrary analytic q-difference modules, one can also obtain in this way a formal classification closely related to Atiyah's classification of holomorphic vector bundles over $\mathbf{E}_q$ (van der Put-Reversat).
In order to get the analytic classification of arbitrary modules (irregular equations), it is necessary (and sufficient) to take in account the canonical slope filtration on the side of the modules; and something called anti-HN filtration (HN stands for Harder-Narasimhan) on the side of the bundles. This is a variant of a statement attributed by Kontsevich and Soibelman to Ramis-Sauloy-Zhang.

Plan

How holomorphic vector bundles come in

The q-world (local analytic view)

The functor $M \rightsquigarrow \mathcal{F}_M$

Slopes and some positive results

The canonical slope filtration

“Anti-HN” filtrations

Added after the talk

Let Σ a compact Riemann surface and let $p : \tilde{\Sigma} \to \Sigma$ its universal covering.

Then every vector bundle $\mathcal{F} \to \Sigma$ can be pulled back to a vector bundle $p^*\mathcal{F} \to \tilde{\Sigma}$

$$\begin{array}{ccc} p^*\mathcal{F} & \longrightarrow & \tilde{\Sigma} \\ \downarrow & & \downarrow p \\ \mathcal{F} & \longrightarrow & \Sigma \end{array}$$

Except if Σ is the Riemann sphere, $\tilde{\Sigma}$ is contractile and therefore $p^*\mathcal{F}$ is trivial:

$$p^*\mathcal{F} = \tilde{\Sigma} \times \mathbf{C}^n.$$

Moreover, $p^*\mathcal{F}$ is *equivariant* under $\mathrm{Aut}(p) = \pi_1(\Sigma)$, *i.e.* every $g \in \mathrm{Aut}(p)$ gives rise to an automorphism of $p^*\mathcal{F}$:

$$(x, X) \mapsto (gx, A(g,x)X), \text{ where } A(g,-) \in \mathrm{GL}_n(\mathcal{O}(\tilde{\Sigma})).$$

This can be used to describe the vector bundle $\mathcal{F}$ as a quotient of $p^*\mathcal{F} = \tilde{\Sigma} \times \mathbf{C}^n$ under the action of $\mathrm{Aut}(p)$:

$$\mathcal{F} = \frac{\tilde{\Sigma} \times \mathbf{C}^n}{(x, X) \sim \text{ all } (gx, A(g,x)X)}.$$

One also gets a description of the sheaf of sections of $\mathcal{F}$ as *a sheaf on* Σ *of functions on* $\tilde{\Sigma}$:

For every open subset U of Σ, the sections of $\mathcal{F}$ over U can be identified with the functions $X : \pi^{-1}(U) \to \mathbf{C}^n$ satisfying the "automorphy" conditions:

$$\forall g \in \mathrm{Aut}(p) \ , \ \forall x \in \pi^{-1}(U) \ , \ X(gx) = A(g,x)X(x).$$

Let $q := e^{2\imath\pi\tau}$, where $\Im \tau > 0$, so that $0 < |q| < 1$.
The map $\mathbf{e} : z \mapsto x := e^{2\imath\pi z}$ induces an isomorphism of elliptic curves and a commutative diagram:

$$\begin{array}{ccc}
\mathbf{C} & \xrightarrow{\quad \mathbf{e} \quad} & \mathbf{C}^* \\
\text{universal covering } p \big\downarrow & & \big\downarrow \pi \text{ (also an etale covering)} \\
\mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau) & \xrightarrow{\ \simeq\ } & \mathbf{C}^*/q^{\mathbf{Z}} =: \mathbf{E}_q
\end{array}$$

If we pull back a *holomorphic* vector bundle (from now on, abreviated bundle):

$$\mathcal{F} \to \mathbf{E}_q$$

along π, we get a bundle over the *open* Riemann surface $\mathbf{C}^*$, and it is therefore trivial:

$$\pi^*\mathcal{F} = \mathbf{C}^* \times \mathbf{C}^n \to \mathbf{C}^*.$$

Of course, $\pi^*\mathcal{F}$ is also equivariant under $\mathrm{Aut}(\pi) = q^{\mathbf{Z}}$.

The equivariant action of the cyclic group $\text{Aut}(\pi) = q^{\mathbf{Z}}$ on the trivial bundle $\pi^*\mathcal{F} = \mathbf{C}^* \times \mathbf{C}^n$ is completely determined by the action of its generator q, which has the form:

$$(x, X) \mapsto (qx, A(x)X), \text{ where } A(x) \in \mathrm{GL}_n(\mathcal{O}(\mathbf{C}^*)).$$

The sheaf on $\mathbf{E}_q$ of sections of the bundle $\mathcal{F}$ can then be described as a sheaf of functions on $\mathbf{C}^*$:

$$\mathcal{F}_A(U) := \{X \in \mathcal{O}(\pi^{-1}(U))^n \mid X(qx) = A(x)X(x)\}$$

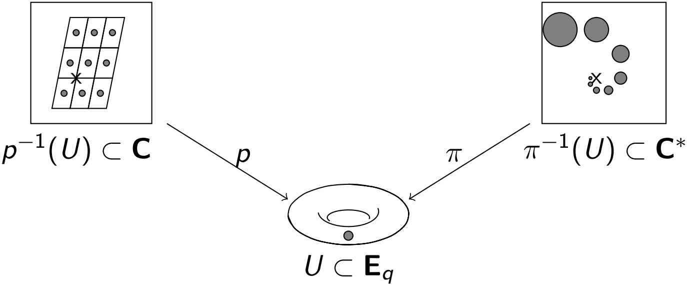

Figure: Preimage in $\mathbf{C}$, resp. in $\mathbf{C}^*$, of a trivializing open set

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So we can see it as *the sheaf of holomorphic solutions of the q-difference equation* $\sigma_q X = AX$*, i.e.* $X(qx) = A(x)X(x)$.

However this $A(x) \in \mathrm{GL}_n(\mathcal{O}(\mathbf{C}^*))$ is "wild" at 0, while in the theory of functional equations the coefficients are usually taken either

- in $\mathbf{C}(x)$ (global study)
- or $\mathbf{C}(\{x\})$ (local analytic study: most of this talk)
- or $\mathbf{C}((x))$ (formal study).

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Recall notations: $\Im\,\tau > 0$, $q := e^{2\imath\pi\tau}$, and $\mathbf{E}_q := \mathbf{C}^*/q^{\mathbf{Z}}$.

Also we denote $K := \mathbf{C}(\{x\}) \subset \hat{K} := \mathbf{C}((x))$ equipped with their q-dilatation automorphism $\sigma_q : f(x) \mapsto f(qx)$.

The Ore-Laurent ring of q-difference operators over K is:

$$\mathcal{D}_{K,q} := K\langle \sigma_q, \sigma_q^{-1} \rangle, \text{ where } \sigma_q.f = \sigma_q(f).\sigma_q$$

A q-difference module (abreviated qdm) over K is defined as a finite length left $\mathcal{D}_{K,q}$-module.

Equivalently, we shall rather see it as a pair (V, Φ) of

- a finite dimensional K-vector space V, equipped with
- a σ_q-linear automorphism Φ of V (*i.e.* a group automorphism such that $\Phi(av) = \sigma_q(a)\Phi(v)$).

Up to the choice of a basis, any qdm can be put in the form:

$M_A := (K^n, \Phi_A)$, where $\Phi_A(X) := A^{-1}\sigma_q X,\ A \in \mathrm{GL}_n(K)$.

(Thus, solutions of $\sigma_q X = AX$ are fixed points of Φ_A.)

We take the qdm $M_A = (K^n, \Phi_A)$ as an algebraic model of the q-difference equation $\sigma_q X = AX$.

The morphisms of qdms $M_A \to M_B$ are the matrices $F \in \mathrm{Mat}_{p,n}(K)$ such that:

$$(\sigma_q F)A = BF.$$

We call $\mathcal{E}_q$ the category of qdms over K. It is abelian, **C**-linear and has a tensor structure (it is tannakian).

The sheaf (on $\mathbf{E}_q$) of solutions (in $\mathbf{C}^*$) of $\sigma_q X = AX$ is:

$$\mathcal{F}_A(U) := \{X \in \mathcal{O}(\pi^{-1}(U), 0)^n \mid \sigma_q X = AX\}.$$

Since A has coefficients in $K = \mathbf{C}(\{x\})$, solutions in $\mathbf{C}^*$ actually have to be *germs* on small open (punctured) neighborhoods of 0 in $\mathbf{C}^*$, *i.e.* in the "germ of space" $(\mathbf{C}^*, 0)$.

The sheaf $\mathcal{F}_A$ is locally free of rank n and corresponds to the bundle (denoted, as is customary, by the same letter):

$$\mathcal{F}_A = \frac{(\mathbf{C}^*, 0) \times \mathbf{C}^n}{(x, X) \sim (x, A(x)X)}.$$

There is an intrinsic definition of the sheaf and bundle $\mathcal{F}_M$ from an arbitrary qdm M, such that $\mathcal{F}_A = \mathcal{F}_{M_A}$.

Plan

Recall that $K = \mathbf{C}(\{x\})$. We set $R := \mathcal{O}(\mathbf{C}^*, 0) \supset K$.

Let $A \in \mathrm{GL}_n(K)$, $B \in \mathrm{GL}_p(K)$ and let:

$$F : M_A = (K^n, \Phi_A) \to M_B = (K^p, \Phi_B)$$

a morphism of qdms, *i.e.* $F \in \mathrm{Mat}_{p,n}(K)$ and $(\sigma_q F)A = BF$.

Now, $A \in \mathrm{GL}_n(R)$, $B \in \mathrm{GL}_p(R)$ and $F \in \mathrm{Mat}_{p,n}(R)$, so that the mapping $X \mapsto FX$ induces a morphism of bundles:

$$\mathcal{F}_A = \frac{(\mathbf{C}^*, 0) \times \mathbf{C}^n}{(x, X) \sim (x, A(x)X)} \to \mathcal{F}_B = \frac{(\mathbf{C}^*, 0) \times \mathbf{C}^p}{(x, X) \sim (x, B(x)X)}.$$

This can be made intrinsic: every $F : M \to N$ in the category $\mathcal{E}_q$ of qdms over K defines a morphism $\mathcal{F}_M \to \mathcal{F}_N$ in the category $\mathcal{V}_q$ of bundles over $\mathbf{E}_q$.

We thereby get a functor $M \rightsquigarrow \mathcal{F}_M$ from $\mathcal{E}_q$ to $\mathcal{V}_q$.

The category $\mathcal{E}_q$ of qdms over K is abelian $\mathbf{C}$-linear (and tannakian).

However, the category $\mathcal{V}_q$ of bundles over $\mathbf{E}_q$ is not abelian (it is only additive).

Therefore we embed $\mathcal{V}_q$ in the category $\mathcal{S}_q$ of coherent sheaves over $\mathbf{E}_q$, which is abelian, $\mathbf{C}$-linear, etc. (Algebraic or analytic sheaves, it amounts to the same.)

The functor $M \rightsquigarrow \mathcal{F}_M$ from $\mathcal{E}_q$ to $\mathcal{S}_q$

It is exact, $\mathbf{C}$-linear, faithful (and $\otimes$-compatible).
Its essential image is $\mathcal{V}_q$.

However, it is *not* fully faithful.

Example 1

Let $n = p = 1$, $A := (x)$ and $B := (1)$.

Morphisms $M_A \to M_B$, resp. $\mathcal{F}_A \to \mathcal{F}_B$, are (scalar) functions $f \in K = \mathbf{C}(\{x\})$, resp. $f \in R = \mathcal{O}(\mathbf{C}^*, 0)$, such that $x\sigma_q f = f$.

There are no such $f \in K \setminus \{0\}$, so $\mathrm{Hom}_{\mathcal{E}_q}(M_A, M_B) = 0$.

But the Jacobi Theta function:

$$\theta_q(x) := \sum_{n \in \mathbf{Z}} q^{n(n-1)/2} x^n \in \mathcal{O}(\mathbf{C}^*) \subset R$$

satisfies $\sigma_q \theta_q = x^{-1} \theta_q$, *i.e.* $\theta_q \in \mathrm{Hom}_{\mathcal{S}_q}(\mathcal{F}_A, \mathcal{F}_B)$.

Therefore $\mathrm{Hom}_{\mathcal{E}_q}(M_A, M_B) = 0$ while $\mathrm{Hom}_{\mathcal{S}_q}(\mathcal{F}_A, \mathcal{F}_B) \neq 0$.

Example 2, courtesy of Julien Roques

Let $n = p = 2$, $A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B := \begin{pmatrix} x^{-1} & 1 \\ 1 & 0 \end{pmatrix}$.
Then one can prove that $\mathrm{Hom}_{\mathcal{E}_q}(M_A, M_B) = 0$.

However, setting:

$$c(x) := \sum_{n \geq 0} \frac{q^{n(n+3)/2}}{(1-q^2)\cdots(1-q^{2n})} x^{-n} \text{ and } d(x) := c(-x),$$

we get an isomorphism $\begin{pmatrix} \sigma_q c & -\sigma_q d \\ c & d \end{pmatrix} : \mathcal{F}_A \to \mathcal{F}_B$ in $\mathcal{S}_q$.

Therefore $\mathrm{Hom}_{\mathcal{E}_q}(M_A, M_B) = 0$ while $\mathcal{F}_A \simeq \mathcal{F}_B$

Functions similar to a, b, c, d appear in Ismaïl-Zhang (2006), as q-analogues of the Airy function.

Plan

To every $L := \sigma_q^n + a_1(x)\sigma_q^{n-1} + \cdots + a_n(x) \in \mathcal{D}_{K,q}$ we associate a *Newton polygon*, the convex hull of the set $\{(i,j) \in \mathbf{Z}^2 \mid j \geq v_x(a_i)\} \subset \mathbf{R}^2$.

The finite part of its boundary is made of k vectors $(r_i, d_i) \in \mathbf{N}^* \times \mathbf{Z}$ with slopes $\mu_i := d_i/r_i \in \mathbf{Q}$ decreasing from right to left: $\mu_1 > \cdots > \mu_k$. Slope μ_i has multiplicity r_i.

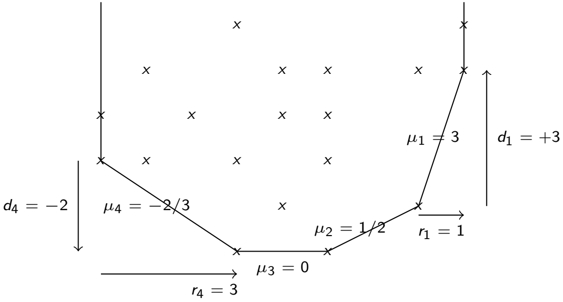

Figure: The Newton polygon

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The Newton polygon is a formal invariant

Given a qdm M, all q-difference operators L such that $M \underset{\hat{K}}{\sim} \mathcal{D}_{K,q}/\mathcal{D}_{K,q}L$ (formal isomorphism) have the same Newton polygon. We write $S(M) := \{\mu_1, \ldots, \mu_k\}$.

We say that M is

- *fuchsian* if $S(M) = \{0\}$,
- *pure isoclinic of slope* μ if $S(M) = \{\mu\}$,
- *pure* if $M = P_1 \oplus \cdots \oplus P_k$, each P_i being pure isoclinic (the slopes "don't mix").

1. Baranovsky-Ginzburg 1996: to a *formal* fuchsian qdm is associated a *flat* vector bundle (*i.e.* one that can be written $\mathcal{F}_A$ with $A \in \mathrm{GL}_n(\mathbf{C})$); this is an equivalence of categories.
   (The equivalence is $\otimes$-compatible from which Kontsevich deduces the formal Galois group of fuchsian qdms in an appendix.)
2. van der Put-Reversat 2007: to any formal qdm is associated a bundle and the formal classification of qdms boils down to Atiyah's classification of bundles over an elliptic curve. This is however not an equivalence of categories, only a bijective correspondence between isomorphy classes.
   (They deduce a complete description of the formal Galois group.)

In both cases, one obtains an *analytic* object (a bundle) from a *formal* one (a qdm over $\hat{K}$) and the construction is rather involved, or at least indirect.

1. JS 1999: $M \rightsquigarrow \mathcal{F}_M$ is an equivalence of fuchsian *analytic* qdms with flat vector bundles (whence the analytic Galois group.)
2. JS 2004: every formal qdm is pure; for pure qdms, the formal and analytic classification are equivalent. (This in some sense "explains" the involved constructions of Baranovsky-Ginzburg and of van der Put-Reversat.)

The 1999 result has as corollary the equivalence of pure isoclinic qdms of a given *integral* slope μ with bundles having the form:

$$\text{(line bundle)} \otimes \text{(flat bundle of slope } \lambda := -\mu).$$

We shall provisionally define:

$$\text{pure bundle } := \text{line bundle} \otimes \text{flat bundle}$$

Plan

The canonical slope filtration

Every qdm M over K admits a unique tower of submodules:

$$M_0 = 0 \subset \cdots \subset M_k = M$$

such that the successive quotients $P_i := M_i/M_{i-1}$ are pure isoclinic of slope μ_i and $\mu_1 < \cdots < \mu_k$. One then has $S(M) = \{\mu_1, \ldots, \mu_k\}$ and the ranks are $r_i = \text{rk } P_i$.
The functor $M \rightsquigarrow \text{gr } M := P_1 \oplus \cdots \oplus P_k$ is exact, **C**-linear, faithful (and $\otimes$-compatible).
Over $\hat{K}$, the filtration splits canonically (*i.e.* the functor gr becomes isomorphic to the identity functor).

Applying the functor $M \rightsquigarrow \mathcal{F}_M$ yields a tower of sub bundles:

$$0 = \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_k = \mathcal{F}_M$$

such that each $\mathcal{F}_i = \mathcal{F}_{M_i}$ and the successive subquotients $\mathcal{G}_i := \mathcal{F}_i/\mathcal{F}_{i-1}$ are the associated bundles $\mathcal{G}_i = \mathcal{F}_{P_i}$.

Case of integral slopes $S(M) \subset \mathbf{Z}$:
Then each P_i can be put into the form $(K^{r_i}, \Phi_{x^{\mu_i} C_i})$ where $C_i \in \mathrm{GL}_{r_i}(\mathbf{C}_i)$.
It follows that the subquotients $\mathcal{G}_i = \mathcal{F}_{x^{\mu_i}} \otimes \mathcal{F}_{C_i}$ are "pure" of slopes $\lambda_i = -\mu_i$ and the slopes *decrease*: $\lambda_1 > \cdots > \lambda_k$.

Case of arbitrary slopes $S(M) \subset r^{-1}\mathbf{Z}$, $r \in \mathbf{N}^*$:
Ramification in the form $x' := x^{1/r}$, $K' := K[x']$, $q' := q^{1/r}$ multiplies the slopes by r so $M' := K' \otimes_K M$ has integral slopes $\mu_i' := r\mu_i \in \mathbf{Z}$.
The corresponding bundle $\mathcal{F}_{M'}$ over $\mathbf{E}_{q'}$ is the pullback $\rho^* \mathcal{F}_M$ by the isogeny $\rho : \mathbf{E}_{q'} \to \mathbf{E}_q$. It has filtration by the $\rho^* \mathcal{F}_i$ with successive quotients $\rho^* \mathcal{G}_i$, which are "pure" of integral slopes $\lambda_i' = r\lambda_i$.

Therefore, in the general case (arbitrary slopes), the quotients $\mathcal{G}_i$ in the filtration of $\mathcal{F}_M$ are the bundles which become "pure" (*i.e.* of the form line bundle $\otimes$ flat bundle) after an isogeny making their slope integral.

We are led to characterize those bundles over an elliptic curve E which become "pure" after pullback by an isogeny $E' \to E$ making their slope integral.

Those are exactly the semi-stable bundles.
(Recall that the bundle $\mathcal{G}$ is *semi-stable* if for every subbundle $\mathcal{G}'$ the slopes satisfy $\lambda(\mathcal{G}') \leq \lambda(\mathcal{G})$.)

The statement follows from two lemmas:

For an bundle with integral slope, "pure" is equivalent to semi-stable.

(Direct implication is easy; the converse uses Weil's characterization of flat bundles.)

If $\rho : E' \to E$ is an isogeny of elliptic curves and if $\mathcal{G}$ is an bundle on E, then $\mathcal{G}$ is semi-stable iff $\rho^*\mathcal{G}$ is.

(Direct implication is by Galois descent and actually much more general; the converse is easy.)

Plan

The filtration on $\mathcal{F}_M$ deduced from the slope filtration on M has successive quotients that are semi-stable with decreasing slopes $\lambda_1 > \cdots > \lambda_k$.
We write $\underline{\mathcal{F}}_M$ the bundle $\mathcal{F}_M$ equipped with that filtration.

This is the opposite behaviour with respect to the so-called *Harder-Narasimhan filtration*, so, following Kontsevich and Soibelman, we call it *anti-HN*.
The definition can be extended to coherent sheaves.

We call $\underline{\mathcal{S}}_q$, resp. $\underline{\mathcal{V}}_q$, the category of coherent sheaves, resp. of bundles, endowed with anti-HN filtrations.

Main result (so far): *local* classification at 0

The functor $M \rightsquigarrow \underline{\mathcal{F}}_M$ from $\mathcal{E}_q$ to $\underline{\mathcal{S}}_q$ is exact, $\mathbf{C}$-linear, *fully* faithful (and $\otimes$-compatible).
Its essential image is $\underline{\mathcal{V}}_q$.

For *global* classification, we should take q-difference modules over $\mathbf{C}(x)$ (*i.e.* $A \in \mathrm{GL}_n(\mathbf{C}(x))$, $F \in \mathrm{Mat}_{p,n}(\mathbf{C}(x))$, etc).

Since $\mathbf{C}(x)$ embeds into $K = \mathbf{C}(\{x\})$, all our constructions remain meaningful, but we should decorate the notations: $\mathcal{F}_M^{(0)}$, $\underline{\mathcal{F}}_M^{(0)}$, $\mathcal{E}_q{}^{(0)}$, etc. at infinity: $\mathcal{F}_M^{(\infty)}$, $\underline{\mathcal{F}}_M^{(\infty)}$, $\mathcal{E}_q{}^{(\infty)}$, etc.

The patching has previously been defined using *Birkhoff's connection matrix*, seen as a *meromorphic isomorphism* $\mathcal{F}_M^{(0)} \to \mathcal{F}_M^{(\infty)}$. Following an idea of Kontsevich-Soibelman, we rather use as patching data the inclusions

$$\mathcal{F}_M^{(0)}, \mathcal{F}_M^{(\infty)}$$

The torsion sheaves $\mathcal{F}_M^{(glob)}/\mathcal{F}_M^{(0)}$ and $\mathcal{F}_M^{(glob)}/\mathcal{F}_M^{(\infty)}$ have been precisely described in a paper by Roques-Sauloy (2019). We hope to soon obtain in this way a satisfying global result.

The q-Riemann-Hilbert correspondence according to Kontsevich-Soibelman

The category of holonomic $\mathcal{D}_q$-modules over $\mathbf{C}^*$ is equivalent to the category of coherent sheaves over $\mathbf{E}_q$ endowed with two anti-HN filtrations labeled by $\mathbf{Q} \cup \{\infty\}$.

- They work over the *ring* $\mathbf{C}[x, x^{-1}]$, whence the need to restrict to *holonomic* modules.
  (The corresponding condition for us is finite length.)
- Probably for the same reason, they have to deal with coherent sheaves instead of vector bundles.
- The label ∞ is here for torsion sheaves, which have infinite slope.
  (In our case, the highest subquotients of the anti-HN filtrations are $\mathcal{F}_M^{(glob)}/\mathcal{F}_M^{(0)}$ and $\mathcal{F}_M^{(glob)}/\mathcal{F}_M^{(\infty)}$.)

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The above is part of a paper in preparation with Julien Roques. Meanwhile, one can look at a draft with the same title on my webpage `http://www.cantoperdic.fr`

The Baranovsky-Ginzburg 1996 article is:
"Conjugacy classes in loop groups and G-bundles on elliptic curves", Internat. Math. Res. Notices 15 (1996), 733–751

The van der Put-Reversat 2007 article is: "Galois theory of q-difference equations", Ann. Fac. Sci. Toulouse Math. (6) 16 (2007), no. 3, 665–718.

The Zhang-Ismaïl 2006 article is:
"Zeros of entire functions and a problem of Ramanujan" Adv. Math. 209, No. 1, 363-380 (2007).

The Roques-Sauloy 2019 article is:
"Euler characteristics and q-difference equations" Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 19, No. 1, 129-154 (2019).

Added after the talk

Work by Kontsevich and Soibelman about the Riemann-Hilbert correspondence can be found in:

https://indico.math.cnrs.fr/category/603/attachments/3705/9402/2025_02_28_Y_Soibelman.pdf

https://www.youtube.com/watch?v=8eesE2SIpZ8

https://www.youtube.com/watch?v=4J_KLTLPwQk

https://www.carmin.tv/fr/collections/maxim-kontsevich/video/riemann-hilbert-correspondence-for-q-difference-modules

Added after the talk

Example 2 (non isomorphic qdms giving rise to isomorphic bundles) was obtained on the basis of the following suggestion of Julien Roques:

JR pour JS, le 27/09/2025, en réponse à “je n'arrive pas à trouver le contre-exemple suivant, dont je pense qu'il existe : deux modules (ou systèmes ou équations) aux q-différences analytiques non isomorphes mais tels que les fibrés associés le soient. Sauf erreur, sous forme concrète, ça veut dire ceci (convention $|q| < 1$). On a $A(x)$ et $B(x)$ dans $\mathrm{GL}_n(\mathbb{C}(x))$ tels qu'il existe F dans $\mathrm{GL}_n(\mathcal{O}(\mathbb{C}^\times, 0))$ (germes sur $\mathbb{C}^\times$ holomorphes au voisinage épointé de 0) avec $F(qx)A(x) = B(x)F(x)$ mais qu'il n'existe pas de tel F dans $\mathrm{GL}_n(\mathbb{C}(x))$. Une méthode serait de partir d'un fibré sur Eq avec deux filtrations anti-HN distinctes, mais je n'arrive pas à en fabriquer.”

Considérons $A(z) \in \mathrm{GL}_n(\mathbb{C}[z, z^{-1}])$ telle que $\sigma_q Y = AY$ soit singulier régulier en 0. Soit $F^{(0)} \in \mathrm{GL}_n(\mathbb{C}(\{z\}))$ une transformation de jauge ramenant $\sigma_q Y = AY$ à un système à coefficients constants $\sigma_q Y = A^{(0)} Y$, $A^{(0)} \in \mathrm{GL}_n(\mathbb{C})$. L'hypothèse $A(z) \in \mathrm{GL}_n(\mathbb{C}[z, z-1])$ implique que $F^{(0)} \in M_n(\mathcal{O}(\mathbb{C}^\times)) \cap \mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$. Puisque la matrice du système dual $\sigma_q Y = A^{-\top} Y$ appartient elle aussi à $\mathrm{GL}_n(\mathbb{C}[z, z-1])$, on a aussi $(F^{(0)})^{-\top} \in M_n(\mathcal{O}(\mathbb{C}^\times)) \cap \mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$. On a donc $F^{(0)} \in \mathrm{GL}_n(\mathcal{O}(\mathbb{C}^\times))$ (et méromorphe en 0, mais ça ne servira pas). Ainsi, le système $\sigma_q Y = AY$ est $\mathrm{GL}_n(\mathcal{O}(\mathbb{C}^\times))$-équivalent à $\sigma_q Y = A^{(0)} Y$. Pour trouver le contre-exemple que tu cherches (en l'infini plutôt qu'en zéro), il suffit donc de trouver un système $\sigma_q Y = AY$ avec $A(z) \in \mathrm{GL}_n(\mathbb{C}[z, z-1])$ qui, en l'infini, n'est pas méromorphiquement équivalent à un système constant ; il suffit pour cela qu'il soit irrégulier en l'infini. Un exemple concret est donné par les systèmes associés aux équations q-hypergéométriques du type $\sigma_q \prod_{i=1}^{n-1} (\sigma_q - a_i) - z$ avec $a_1, \ldots, a_{n-1} \in \mathbb{C}$.

Added after the talk

First question asked at the end of the talk: “Is there a conjectural description of the global Galois group ?”

Answer: There is a complete description (Sauloy) in the fuchsian case in:
“Galois theory of Fuchsian q–difference equations” Ann. Sci. Éc. Norm. Supér. (4) 36, No. 6, 925-968 (2003).

There is a complete description (Ramis-Sauloy) in the irregular case assuming integral slopes in: “The q–analogue of the wild fundamental group and the inverse problem of the Galois theory of q-difference equations” Ann. Sci. Éc. Norm. Supér. (4) 48, No. 1, 171-226 (2015).

There is a partial description (which will probably be completed into a complete description in the final publication) in the irregular case allowing arbitrary slopes in: arXiv:2006.03237

Added after the talk

Second question asked at the end of the talk: "Does the functor preserve the Ext ?"

Answer: I think it does. See 4.4.3 in the text "qRHB+HVB" on my webpage for a very particular instance. The matter will be settled in the final version of the paper.

For any additional question, please write me:
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