Classification of irregular q-difference equations using holomorphic vector bundles

Jacques Sauloy
jacquessauloy@gmail.com
http://www.cantoperdic.fr/

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0 Introduction

This text is part of a common work with Julien Roques and partly inspired by work of Kontsevich and Soibelman.

0.1 Some general conventions and notations

À ÉLAGUER, TOUT ÇA NE SERT PAS ICI

Among the notations of general use, let us mention: the subgroup μ_r of r^{th} roots of unity and the group $\mu_\infty := \bigcup_{r\geq 1} \mu_r$ of all roots of unity in $\mathbf{C}^*$; the *base* $q \in \mathbf{C}^*$ (see just below); the elliptic curve $\mathbf{E}_q := \mathbf{C}^*/q^{\mathbf{Z}}$; and the fundamental annulus $\mathcal{C}_q := \{z \in \mathbf{C} \mid |q| < |z| \leq 1\}$ (so the natural projection $\pi : \mathbf{C}^* \to \mathbf{E}_q$ induces a bijection from $\mathcal{C}_q$ to $\mathbf{E}_q$). We also let $\mathrm{VM}(a_1, \ldots, a_n)$ the Vandermonde matrix $(a_i^j)_{\substack{i=1,\ldots,n \\ j=0,\ldots,n-1}}$ and $\mathrm{VM}'(a_1, \ldots, a_n)$ its transpose.

0.1.1 Base and ramification: notations

In all the text, q is a complex number such that $0 < |q| < 1$. We select τ such that $q = e^{2\mathrm{i}\pi\tau}$ (so τ belongs to the Poincaré half plane $\mathcal{H} : \Im\, \tau > 0$). Then non integral powers[1] of q are well defined as $q^\alpha := e^{2\mathrm{i}\pi\tau\alpha}$.

Note however that, although most of the study is "relative" (we replace many times the *base* q by some q_r, in order to have integral slopes) there is all along an "absolute" base q. We hope someday to be able to work in a more consistently relative setting.

Let $K := \mathbf{C}((x))$ or $\mathbf{C}(\{x\})$ made a difference field (K, σ) when endowed with the automorphism $\sigma := \sigma_q : f(x) \mapsto f(qx)$. Each time we resort to *ramification* at a "level" $n \in \mathbf{N}^*$, we let:

$$j := e^{2\mathrm{i}\pi/n}, \quad q' := q^{1/n}, \quad x' := x^{1/n}, \quad K' := K[x'] = \mathbf{C}((x')) \text{ or } \mathbf{C}(\{x'\}),$$

so K'/K is a cyclic extension of K with Galois group $\mathrm{Gal}(K'/K) \simeq \mu_n$ generated by the automorphism $\tau : f(x') \mapsto f(jx')$; and the automorphism $\sigma' := \sigma_{q'} : f(x') \mapsto f(q'x')$ makes (K', σ') a q-difference extension of (K, σ).

Also we denote $\mathcal{D}$ the ring $\mathcal{D}_{K,q}$ of q-difference operators over K and $\mathcal{D}' := K' \otimes_K \mathcal{D}$ its natural extension over K' [16, 8.3.3].

0.1.2 The matricial model (K^n, Φ_A)

Consider $K := \mathbf{C}(\{x\})$ or $\mathbf{C}((x))$ as a difference field by endowing it with the automorphism $\sigma_q : f(x) \mapsto f(qx)$. A q-difference module over K is a finite dimensional K-linear space V endowed with a σ_q-linear automorphism Φ, *i.e.* Φ is a group automorphism such that $\Phi(av) = \sigma_q(a)\Phi(v)$ for every $a \in K$, $v \in V$.

For example, if $A \in \mathrm{GL}_n(K)$, setting $\Psi_A(X) := A\sigma_q X$, we obtain a q-difference module (K^n, Ψ_A) (of course, the Φ_A of our previous standard convention is $\Psi_{A^{-1}}$). Actually, every q-difference module (V, Φ) is isomorphic to

[1] Actually, all that is really needed is a consistent family of r^{th} roots $q_r \in \sqrt[r]{q}$, $r \in \mathbf{N}^*$, *i.e.* one such that $q_1 = q$ and $q_{rs}^s = q_r$. Note that we presently do *not* know what are the consequences of the choice of a particular logarithm τ (or of a particular family of r^{th} roots q_r). This is an interesting enigma which certainly deserves further thought.

some (K^n, Ψ_A): choosing a basis $\mathcal{B}$ of V over K, from the proof of Proposition 8.8 in [16] we see that $\Phi(\mathcal{B})$ also is a basis, and writing it as $\mathcal{B}A$ entails that $X \mapsto \mathcal{B}X$ is an isomorphism of q-difference modules $(K^n, \Psi_A) \to (V, \Phi)$.

A morphism of q-difference modules $(V, \Phi) \to (V', \Phi')$ is a K-linear map $F : V \to V'$ such that $\Phi' \circ F = F \circ \Phi$. In the matricial model, a morphism $(K^n, \Psi_A) \to (K^m, \Psi_B)$ is a matrix $F \in \mathrm{Mat}_{n,m}(K)$ such that $FA = B\sigma_q F$. Therefore the corresponding notion of *gauge transform* of $A \in \mathrm{GL}_n(K)$ by $F \in \mathrm{GL}_n(K)$ (see [16, Subsubsection 7.1.2.2] for comparison) is $B := FA(\sigma_q F)^{-1}$, which ensures that $n = m$ and $(K^n, \Psi_A) \simeq (K^m, \Psi_B)$ (and, of course, conversely).

As was the case for the (K^n, Φ_A) model, ramification does not change the matrix; with obvious meaning of the notations:

$$K' \otimes (K^n, \Psi_A) = (K'^n, \Psi'_A), \text{ where } \Psi'_A : X \mapsto A\sigma' X.$$

Last, the tensor product is obtained as:

$$(K^n, \Psi_A) \otimes (K^m, \Psi_B) = (K^{nm}, \Psi_{A \otimes B}).$$

0.1.3 Fields, categories, ...

Unifier les notations afin de rendre vrai ce qui suit.

We list the notations for the various fields, categories and other various fundamental objects appearing here.

1. $\mathbf{E}_q$ denotes $\mathbf{C}^*/q^{\mathbf{Z}}$ seen either as a group, a Riemann surface, or an elliptic curve.
2. $K := \mathbf{C}(\{x\})$ and $\hat{K} := \mathbf{C}((x))$ denote the fields of convergent, resp. formal, Laurent power series.
3. $K_r := K[x_r]$ and $\hat{K}_r := \hat{K}[x_r]$ (with $x_r := x^{1/r}$) denote their respective unique cyclic extension of degree r; we often write them $K' := K_r$, $\hat{K}' := \hat{K}_r$ when r is fixed. They are endowed with the $q' = q_r$-dilatation operator, where $q_r := q^{1/r}$, making them q-difference extensions of $K, \hat{K}$.
4. $\mathscr{E}_q$, resp. $\hat{\mathscr{E}}_q$, denotes the category of analytic, resp. formal, q-difference modules, *i.e.* over K, resp. over $\hat{K}$; and $\mathcal{P}_q$ the full subcategory of $\mathscr{E}_q$ with objects the pure q-difference modules.
5. $\mathscr{S}_q$ denotes the category of coherent sheaves over $\mathbf{E}_q$ and $\mathscr{V}_q$ its full subcategory with objects the locally free sheaves, *aka* holomorphic vector bundles.

0.2 Some basic techniques

0.2.1 Base change (extension and restriction of scalars) for q-difference modules

We recall (and complete) from [16, Section 8.3] the change of base fields for difference modules. So we let (K, σ) a difference field and (K', σ') a difference extension of (K, σ) and $\mathscr{C}, \mathscr{C}'$ the corresponding categories of difference modules.

For every $M = (V, \Phi)$ in $\mathscr{C}$, we define $K' \otimes_K M := (K' \otimes_K V, \Phi')$ in $\mathscr{C}'$, where Φ' is the unique σ'-linear automorphism of $K' \otimes_K V$ such that $a \otimes v \mapsto \sigma'(a) \otimes \Phi(v)$. Conversely, for every $N' = (W', \Psi')$ in $\mathscr{C}'$, we define $N'_{(K)} := (W, \Psi)$ in $\mathscr{C}$ by setting $W := W'_{(K)}$ (restriction of scalars) and $\Psi := \Psi'$. Those operations are not inverse of each other (composing them in either sense multiplies the rank by $[K' : K]$) but there is an adjunction formula [16, Subsubsection 8.3.4.1]:

$$\mathrm{Hom}_{\mathscr{C}}(M, N'_{(K)}) = \mathrm{Hom}_{\mathscr{C}'}(K' \otimes_K M, N'). \tag{1}$$

If $f : M \to N'_{(K)}$ is a morphism in $\mathscr{C}$, then the corresponding $f' : K' \otimes_K M \to N'$ is defined by $a \otimes v \mapsto af(v)$; and conversely any $f' : K' \otimes_K M \to N'$ defines f as its restriction to V, identified to $1 \otimes V \subset V'$. Note that:

$$\big(f = 0 \iff f' = 0\big) \text{ and } \big(f' \text{ injective } \implies f \text{ injective}\big).$$

Extension and restriction of scalars preserve inclusion and quotients (for extension, this is because we deal with vector spaces; for restriction, see [4, II §1 no 13]). We shall use those facts as follows. Let first $0 = M_0 \subset \cdots \subset M_k = M$ a tower of submodules in $\mathscr{C}$. Then setting $M'_i := K' \otimes_K M_i$ and $M' := K' \otimes_K M$, we get a tower of submodules $0 = M'_0 \subset \cdots \subset M'_k = M'$ in $\mathscr{C}'$. Moreover:

$$K' \otimes \frac{M_i}{M_{i-1}} = \frac{K' \otimes M_i}{K' \otimes M_{i-1}}.$$

Now let $0 = N'_0 \subset \cdots \subset N'_l = N'$ a tower of submodules in $\mathscr{C}'$. Then setting $N_j := (N'_j)_{(K)}$ and $N := N'_{(K)}$, we get a tower of submodules $0 = N_0 \subset \cdots \subset N_l = N$ in $\mathscr{C}$. Moreover:

$$\left(\frac{N'_j}{N'_{j-1}}\right)_{(K)} = \frac{(N'_j)_{(K)}}{(N'_{j-1})_{(K)}}.$$

A useful particular case of the adjunction relation (1) is obtained by taking $N' := K' \otimes M$. Then $f' = \mathrm{Id}_{N'}$ on the right hand side corresponds to some $f : M \to (K' \otimes M)_{(K)}$ on the left hand side. By the previous remark that f' injective implies f injective, this makes M a submodule of $(K' \otimes M)_{(K)}$ in $\mathscr{C}$.

0.2.2 Extension of scalars followed by restriction

In the case that K'/K is a finite cyclic extension, the injection of M into $(K' \otimes M)_{(K)}$ takes a special form. We use ramification and Galois descent.

Lemma 0.1. *That injection makes M a direct summand of $M'_{(K)}$.*

Proof. Let $r := [K' : K]$ and $p := \dfrac{1}{r} \sum_{i=0}^{r-1} T^i$. Then p is K-linear (because T is) and:

$$pT = Tp = \frac{1}{r} \sum_{i=1}^{r} T^i = p \implies p^2 = \frac{1}{r} \sum_{i=1}^{r} pT^i = \frac{1}{r} \sum_{i=1}^{r} p = p.$$

Thus p is a projector of $V'_{(K)}$. It is the identity when restricted to ${V'}^T = V$ and its image is clearly included in ${V'}^T$, so it is exactly V.

On the other hand, p commutes with Φ' (since T does) and since the structural σ-linear automorphism of $M'_{(K)}$ is Φ' itself, the image of p is (the underlying space of) a sub-q-difference module; this is, of course, the natural injection of M in $M'_{(K)}$. But the same applies to the kernel of p, so indeed M is a direct factor of the q-difference module M'. □

0.2.3 Formal *vs* pure analytic q-difference modules

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We shall denote $\mathscr{E}_q$, resp. $\hat{\mathscr{E}}_q$ the category of q-difference modules over $K := \mathbf{C}(\{x\})$, resp. over $\hat{K} := \mathbf{C}((x))$; and we shall respectively speak of *analytic*, resp. of *formal* q-difference modules. The full subcategory of $\mathscr{E}_q$ with objects the pure analytic q-difference modules is denoted $\mathcal{P}_q$.

We intend to make precise here the fact that the formal classification of q-difference modules (be they formal or analytic) amounts to the same as the analytic classification of pure analytic q-difference modules. First recall some elementary facts from [16, Chapter 10].

1. (Theorem 10.4) The functor gr from $\mathscr{E}_q$ to $\mathcal{P}_q$ is additive, $\mathbf{C}$-linear, exact, faithful and $\otimes$-compatible. It is a retraction of the inclusion of $\mathcal{P}_q$ into $\mathscr{E}_q$.

2. (Corollary 10.5) Here we denote $\hat{\text{gr}}$ the graduation functor for formal q-difference modules; then the functor $\hat{\text{gr}}$ from $\hat{\mathscr{E}}_q$ to itself is isomorphic to the identity functor.

3. (Theorem 10.5) Let M, N to objects in $\mathscr{E}_q$; then:

$$\hat{M} \simeq \hat{N} \text{ (in } \hat{\mathscr{E}}_q) \Longleftrightarrow \text{gr } M \simeq \text{gr} N \text{ (in } \mathscr{E}_q).$$

4. (Lemma 10.53) Let $A, B \in \mathrm{GL}_n(K)$ matrices of pure isoclinic analytic q-difference modules having the same slope and let $F \in \mathrm{GL}_n(\hat{K})$ such that $F[A] = B$. Then $F \in \mathrm{GL}_n(K)$.

It was asked in [16, Exercise 10.54] to generalize Lemma 10.53 to all morphisms, which we now do.

Lemma 0.2. *Let $A \in GL_n(K)$, $B \in GL_p(K)$ matrices of pure isoclinic analytic q-difference modules having the same slope and let $F \in Mat_{p,n}(\hat{K})$ a formal morphism from A to B,* i.e. *$(\sigma_q F)A = BF$. Then $F \in Mat_{p,n}(K)$.*

Proof. Equation $(\sigma_q F)A = BF$ is equivalent to $\sigma_q F = BFA^{-1}$, which characterizes solutions of $M_A^\vee \otimes M_B = \underline{\mathrm{Hom}}(M_A, M_B)$; but the latter "internal Hom" q-difference module is pure isoclinic of slope 0, *i.e.* fuchsian. So it all boils down to the fact that formal solutions of fuchsian q-difference modules are analytic, see the next lemma. □

Lemma 0.3. *let $A \in GL_n(K)$ and let $X \in \hat{K}^n$ such that $\sigma_q X = AX$. Then $X \in K^n$.*

Proof. By [16, Corollary 11.47], we can write $A = F[B]$ with $F \in \mathrm{GL}_n(K)$ and $B \in \mathrm{GL}_n(\mathbf{C})$. Then, setting $X = FY$ we see that: $\sigma_q X = AX$ is equivalent to $\sigma_q Y = BY$; that the assumption $X \in \hat{K}^n$ is equivalent to $Y \in \hat{K}^n$; and that the conclusion $X \in K^n$ is equivalent to $Y \in K^n$. So we may as well assume from start that $A \in \mathrm{GL}_n(\mathbf{C})$.
In that case; expanding X as a formal Laurent series $\sum X_m x^m$, we see that $\sigma_q X = AX$ is equivalent to: $q^m X_m = AX_m$ for all m. But the latter equation admits a non zero solution if, and only if, q^m is an eigenvalue of A, that is only for a finite number of values of m. Therefore in that case we find that $X \in \mathbf{C}[x, x^{-1}]^n$. □

Proposition 0.4. *The functor $\mathcal{P}_q \to \hat{\mathscr{E}}_q$ induced by the formalisation functor $\mathscr{E}_q \to \hat{\mathscr{E}}_q$ is fully faithful.*

Proof. Let $M = \bigoplus M^{(\mu)}$ and $N = \bigoplus N^{(\mu)}$ pure analytic modules decomposed into their pure isoclinic components and let $\hat{M}, \hat{N}$ their formalisations, so that $\hat{M} = \bigoplus \hat{M}^{(\mu)}$ and $\hat{N} = \bigoplus \hat{N}^{(\mu)}$. Morphisms from $\hat{M}$ to $\hat{N}$ decompose into: morphisms from $\hat{M}^{(\mu)}$ to $\hat{N}^{(\nu)}$ with $\mu \neq \nu$, which must be 0; and morphisms from $\hat{M}^{(\mu)}$ to $\hat{N}^{(\mu)}$ which must be defined over K by the previous lemma. □

That functor is actually an equivalence of tensor categories.

1 Local q-Riemann-Hilbert-Birkhoff correspondence through holomorphic vector bundles over $\mathbf{E}_q$

For basic general information on holomorphic vector bundles over an elliptic curve, see Appendix A. We shall use the abbreviation HVB(s) for holomorphic vector bundle(s).

In [16], we associated to every q-difference module over $\mathbf{C}(\{x\})$ a holomorphic vector bundle as follows. Let for instance$M = M_A := (\mathbf{C}(\{x\})^n, \Phi_A)$, where $A \in \mathrm{GL}_n(\mathbf{C}(\{x\}))$. Then there is a locally free sheaf of solutions sending every open subset $V \subset \mathbf{E}_q$ to[2]:

$$\mathscr{F}_A(V) := \{X \in \mathscr{O}_{\mathbf{C}^*}\left(\pi^{-1}(V), 0\right)^n \mid \sigma_q X = AX\}. \tag{2}$$

(Recall that $\pi : \mathbf{C}^* \to \mathbf{E}_q$ is the canonical quotient map.) Every locally free sheaf can be identified to the sheaf of sections of a HVB, traditionally denoted by the same letter. In the present case that HVB can be described geometrically:

$$\mathscr{F}_A := \frac{(\mathbf{C}^*, 0) \times \mathbf{C}^n}{(x, X) \sim (qx, A(x)X)}. \tag{3}$$

We shall soon (Subsubsection 1.1.1) give details on the meaning and on the use of the "germ of space" $(\mathbf{C}^*, 0)$. On the other hand, every HVB over $\mathbf{E}_q$ can be described in a totally similar way, starting from a matrix $A \in \mathrm{GL}_n(\mathscr{O}(\mathbf{C}^*))$: such a matrix being given, equation (3) defines a HVB and equation (2) describes its sheaf of sections (see Subsection A.2 of Appendix A).

Obviously there is a kinship between both descriptions; and it is actually possible to describe the local analytic classification of q-difference modules over $\mathbf{C}(\{x\})$ in terms of HVBs with some additional structure related to the canonical slope filtration.

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So in this section we shall first detail the construction of the HVB $\mathscr{F}_A$ in a setting that subsumes both the case $A \in \mathbf{GL}_n(\mathbf{C}(\{x\}))$ and the case $A \in \mathbf{GL}_n(\mathscr{O}(\mathbf{C}^*))$: that is, starting from a matrix $A \in \mathbf{GL}_n(\mathscr{O}(\mathbf{C}^*, 0))$.

Then we shall describe the properties of the functor which sends a q-difference module to a HVB; to go backwards, that is from a matrix in $\mathbf{GL}_n(\mathscr{O}(\mathbf{C}^*, 0))$ to one in $\mathbf{GL}_n(\mathbf{C}(\{x\}))$, will involve some kinds of "regularization" procedures.

Last, we shall see how to enrich that functor so as to obtain a complete local analytic classification of analytic q-difference modules.

1.1 From analytic q-difference modules to HVBs over $\mathbf{E}_q$

1.1.1 HVB defined from a matrix $A \in \mathbf{GL}_n(\mathscr{O}(\mathbf{C}^*, 0))$

The following constructions are easy generalisations of those described in Subsection A.2 of Appendix A for a matrix $A \in \mathrm{GL}_n(\mathscr{O}(\mathbf{C}^*))$ and we shall leave it to the reader to fill in the necessary arguments.

[2]Since all studies here are local at 0 we here denote $\mathscr{F}_A$ what was denoted $\mathscr{F}_A^{(0)}$ in [16].

Let $A \in \mathrm{GL}_n(\mathscr{O}(\mathbf{C}^*,0))$. Let $\overset{\bullet}{\Delta} := \overset{\bullet}{\mathrm{D}}(0,r)$ a punctured open disk of center 0 and radius $r>0$ such that A is holomorphic and regular over $\overset{\bullet}{\Delta}$ (this means that $A \in \mathrm{GL}_n(\mathscr{O}(\overset{\bullet}{\Delta}))$). The action $(q^k,x) \mapsto q^k x$ of the semigroup $q^{\mathbf{N}}$ on $\overset{\bullet}{\Delta}$ by q-dilatations extends to an equivariant action on the trivial HVB $\overset{\bullet}{\Delta} \times \mathbf{C}^n$, uniquely determined by the action of the generator q:

$$q.(x,X) := (qx, A(x)X).$$

It follows that $q^k.(x,X) = (q^k x, A_k(x)X)$ for arbitrary $k \in \mathbf{N}$, where:

$$A_k := (\sigma_q^{k-1}A)\cdots(\sigma_q A)A.$$

Then the quotient of $\overset{\bullet}{\Delta} \times \mathbf{C}^n$ by this action is a HVB over the Riemann surface $\overset{\bullet}{\Delta}/q^{\mathbf{N}}$, which itself may be identified with $\mathbf{E}_q$. Let us write $\mathscr{F}_{A,\overset{\bullet}{\Delta}} \to \mathbf{E}_q$ that HVB. Now if $0<r'<r$ and $\overset{\bullet}{\Delta}' := \overset{\bullet}{\mathrm{D}}(0,r')$, so that $\overset{\bullet}{\Delta}' \subset \overset{\bullet}{\Delta}$, the inclusion $\overset{\bullet}{\Delta}' \times \mathbf{C}^n \subset \overset{\bullet}{\Delta} \times \mathbf{C}^n$ is compatible with the actions of $q^{\mathbf{N}}$ and we obtain a morphism of HVBs $\mathscr{F}_{A,\overset{\bullet}{\Delta}'} \to \mathscr{F}_{A,\overset{\bullet}{\Delta}}$. *This is actually an isomorphism*, and the direct system of all such isomorphisms (indexed by the directed set of all relevant punctured disks) canonically defines (as its direct limit) a HVB which we call $\mathscr{F}_A$ and describe geometrically by equation (3); and equation (2) then describes the associated sheaf of sections.

Then, exactly as in the special cases of matrices in $\mathrm{GL}_n(\mathscr{O}(\mathbf{C}^*))$ (Subsection A.2) we see that, if $A \in \mathrm{GL}_n(\mathscr{O}(\mathbf{C}^*,0))$ and $B \in \mathrm{GL}_p(\mathscr{O}(\mathbf{C}^*,0))$, then morphisms of HVBs $\mathscr{F}_A \to \mathscr{F}_B$ are induced by rectangular matrices $F \in \mathrm{Mat}_{p,n}(\mathscr{O}(\mathbf{C}^*,0))$ such that $(\sigma_q F)A = BF$.

As useful tools expressed in that formalism, let us describe line bundles and short exact sequences, starting with the latter. If $A \in \mathrm{GL}_n(\mathscr{O}(\mathbf{C}^*,0))$ and $B \in \mathrm{GL}_p(\mathscr{O}(\mathbf{C}^*,0))$, then every exact sequence $0 \to \mathscr{F}_A \to \mathscr{F} \to \mathscr{F}_B \to 0$ is isomorphic to the natural exact sequence $0 \to \mathscr{F}_A \to \mathscr{F}_C \to \mathscr{F}_B \to 0$, where $C := \begin{pmatrix} A & U \\ 0 & B \end{pmatrix}$ for some $U \in \mathrm{Mat}_{n,p}(\mathscr{O}(\mathbf{C}^*,0))$.

As for line bundles, they obviously are the $\mathscr{F}_u$, $u \in \mathscr{O}(\mathbf{C}^*,0)^*$. This can be "regularized" as follows:

Lemma 1.1. *Let $\mathscr{L}$ a line bundle over $\mathbf{E}_q$. Then there exists $c \in \mathbf{C}^*$ and $k \in \mathbf{Z}$ such that $\mathscr{L}$ is isomorphic to $\mathscr{F}_{cx^k}$.*

Proof. There are two possible proofs. In the spirit of vector bundle theory, we note that $\mathscr{L} = \mathscr{O}_{\mathbf{E}_q}(D)$, where D is the divisor of any of its meromorphic sections. So we just have to exhibit, for any such D, a function $f \in \mathscr{M}(\mathbf{C}^*)^*$ such that $\mathrm{div}_{\mathbf{E}_q}(f) = D$ and $\sigma_q f = cx^k f$ for some $c \in \mathbf{C}^*$, $k \in \mathbf{Z}$. We let $D = \sum n_i[\overline{c_i}]$ and choose:

$$f := \prod \theta_q(-x/c_i)^{n_i}, \text{ so that } c = \prod(-c_i)^{n_i} \text{ and } k = -\sum n_i.$$

In the spirit of q-difference module theory, we first write $\mathscr{L}$ as $\mathscr{F}_u$, $u \in \mathscr{O}(\mathbf{C}^*)^*$ and we "regularize" u, *i.e.* we find $f \in \mathbf{C}(\{x\})^*$ and $c \in \mathbf{C}^*$, $k \in \mathbf{Z}$ such that

$u = cx^k(\sigma_q f)/f$, so that f is an isomorphism of HVBs $\mathscr{F}_{cx^k} \simeq \mathscr{L}$. To do that, we first write $xu'/u \in \mathscr{O}(\mathbf{C}^*)$ as $\sum a_n x^n$. By Cauchy's Residue Theorem applied to u'/u, we see that $a_0 \in \mathbf{Z}$, this is k. Then we let:

$$g := \sum_{n \neq 0} \frac{a_n}{n(q^n - 1)} \in \mathscr{O}(\mathbf{C}^*), \text{ so that } \sigma_q(xg') - (xg') = xu'/u - k,$$

(we use the fact that the fact that the operator $f \mapsto xf'/f$ commutes with σ_q) and we can take $f := e^g$. (Yet another proof will be given at the beginning of Subsubsection 1.2.1.) □

1.1.2 Some "regularization" procedures

In Subsubsection 1.1.3 we shall have use for the following fact. It is stated for $\mathscr{O}(\mathbf{C}^*, 0)$, but the same proof works for $\mathscr{O}(\mathbf{C}^*)$ yielding the same conclusion (see Corollary 1.5 further below).

Proposition 1.2. *Let $c \in \mathbf{C}^*$ and $k \in \mathbf{Z}$. Let $\Phi_{c,k} : \mathscr{O}(\mathbf{C}^*, 0) \mapsto \mathscr{O}(\mathbf{C}^*, 0)$ the $\mathbf{C}$-linear map $f \mapsto cx^k \sigma_q f - f$. Then:*

$$\mathscr{O}(\mathbf{C}^*, 0) = Im\ \Phi_{c,k} \oplus \begin{cases} \mathbf{C} \oplus \cdots \oplus \mathbf{C}x^{|k|-1} \text{ if } k < 0, \\ \mathbf{C}x^{-m} \text{ if } k = 0, c = q^m, \\ \{0\} \text{ if } (k = 0, c \notin q^{\mathbf{Z}}) \text{ or } k > 0. \end{cases}$$

Proof. If $k < 0$, this is a particular case of [15, Lemma 2.9] (where it appears in the course of the proof). If $k = 0$, whether or not $c \in q^{\mathbf{Z}}$, it immediately stems from the following calculation (with obvious Laurent series notations for elements of $\mathscr{O}(\mathbf{C}^*)$):

$$c\sigma_q f - f = g \Longleftrightarrow \forall n\ ,\ (cq^n - 1)f_n = g_n,$$

so either no $cq^n - 1$ vanishes and there is a unique f, or one vanishes and the corresponding term of g must be isolated.
Now let $k > 0$ and write $g \in \mathscr{O}(\mathbf{C}^*)$ as $g_1 + g_2$, where $g_1 \in \mathbf{C}(\{x\})$ and $g_2(x) = \gamma(y)$, $y := 1/x$, $\gamma \in \mathbf{C}(\{y\})$. We solve separately $cx^k\sigma_q f_i - f_i = g_i$, $i = 1, 2$. For the first one we apply Lemma 1.4 herebelow. For the second one, we set $f_2(x) = \phi(y)$ and we argue:

$$\begin{aligned} cx^k\sigma_q f_2 - f_2 = g_2 &\Longleftrightarrow cy^{-k}\phi(y/q) - \phi(y) = \gamma(y) \\ &\Longleftrightarrow (q^k/c)y^k\phi(qy) - \phi(y) = (-q^k/c)y^k\gamma(qy) \end{aligned}$$

which also admits a solution $\phi \in \mathscr{O}(\mathbf{C})$ by the same lemma. □

Remark 1.3. Note that we really need that $\phi \in \mathscr{O}(\mathbf{C})$ so that f_2 converges in a punctured neighborhood of 0.

Lemma 1.4. *Let $c \in \mathbf{C}^*$, $k \in \mathbf{Z}$, $k > 0$ and let $\Phi_{c,k}$ as in Proposition 1.2. Then, for every $R > 0$, the map $\Phi_{c,k}$ from $\mathscr{O}(D(0, R))$ to itself is onto.*

Proof. Let $r \in]0; R[$ such that $\alpha := |c|\, r^k < 1$. For $f \in \mathscr{O}(D(0, r)) \cap \mathscr{C}(\overline{\mathrm{D}}(0, r))$, write $Lf := cx^k\sigma_q f$. Then, using the max-norm, $\|Lf\| \leq \alpha \|f\|$, *i.e.* L is a contraction. So the equation $\Phi_{k,c}(f) = g$, which amounts to $f = -g + Lf$, can be solved by iteration with $f \in \mathscr{O}(D(0, r))$.
Now, for every $r < R$, we can extend the domain of f to $D(0, r')$, where $r' := \min(R, r/|q|)$ we just use for that the functional equation $f = -g + Lf$. □

Corollary 1.5. *Proposition 1.2 holds true if $\mathscr{O}(\mathbf{C}^*, 0)$ is replaced by $\mathscr{O}(\mathbf{C}^*)$.*

Proof. In the proof of Lemma 1.4, the last part of the argument shows that f can be taken in $\mathscr{O}(D(0, R/|q|))$, as is usual with such "regularizing" functional equations [15]. Therefore the punctured disk of convergence may be enlarged at will. □

We shall provisionally denote $R_{c,k}(g)$ a pair (f, u) such that $g = \Phi_{c,k}(f) + u$ in the decomposition obtained in Proposition 1.2; such an u is of course unique, but, if Ker $\Phi_{c,k} \neq \{0\}$ (which occurs precisely when $k = 0$, $c \in q^{\mathbf{Z}}$), f may have to be *chosen* (and that, with some arbitrariness).

1.1.3 HVBs associated to q-difference modules

In matricial form, a q-difference module can be described as:

$$M = M_A := (\mathbf{C}(\{x\})^n, \Phi_A), \text{ where } A \in \mathrm{GL}_n(\mathbf{C}(\{x\})) \text{ and } \Phi_A : X \mapsto A^{-1}\sigma_q X.$$

Then equations (3) and (2) show us how to define the HVB and the locally free sheaf both denoted $\mathscr{F}_A$.

In more intrinsic terms, $M = (V, \Phi)$ where V is a finite dimensional vector space over $\mathbf{C}(\{x\})$ and Φ a σ_q-linear automorphism of V. Then by [16, 9.3] one can still define $\mathscr{F}_M$. A morphism of q-difference modules $M \to N$ directly yields a morphism $\mathscr{F}_M \to \mathscr{F}_N$ of HVBs and we have a functor from the category $\mathscr{E}_q$ of q-difference modules over $\mathbf{C}(\{x\})$ to the category $\mathscr{V}_q$ of HVBs over $\mathbf{E}_q$.

To speak of the abelian properties of that functor, we introduce the category $\mathscr{S}_q$ of coherent sheaves over $\mathbf{E}_q$, an abelian tensor category, of which $\mathscr{V}_q$ is a full subcategory. However, $\mathscr{V}_q$ is an additive category, not an abelian one.

Proposition 1.6. *(i) The functor $M \rightsquigarrow \mathscr{F}_M$ from $\mathscr{E}_q$ to $\mathscr{S}_q$ is additive, $\mathbf{C}$-linear, exact, faithful and $\otimes$-compatible.*
(ii) Its essential image is $\mathscr{V}_q$.

Proof. (i) See [16, 9.3].
(ii) We already know that every image $\mathscr{F}_M$ is an object of $\mathscr{V}_q$ (an HVB). Conversely, we must show that every HVB over $\mathbf{E}_q$ is isomorphic to $\mathscr{F}_M$ for some q-difference module M over $\mathbf{C}(\{x\})$. According to the matricial descriptions discussed above, this means that:

$$\forall B \in \mathrm{GL}_n(\mathscr{O}(\mathbf{C}^*)) \ , \ \exists A \in \mathrm{GL}_n(\mathbf{C}(\{x\})) \text{ and } F \in \mathrm{GL}_n(\mathscr{O}(\mathbf{C}^*, 0)) \ : \ B = F[A],$$

where the *gauge transform* $F[A]$ is defined to be $(\sigma_q F)AF^{-1}$. So in some sense this is (again) a "regularization" statement, where a wild singularity at 0 is "tamed". The proof goes in two steps.
First, in Lemma 1.1 we have seen that every line bundle over $\mathbf{E}_q$ is isomorphic to some $\mathscr{F}_u$ with $u \in \mathbf{C}(\{x\})^*$, actually $u = cx^k$ for some $c \in \mathbf{C}^*$, $k \in \mathbf{Z}$.
Let now $\mathscr{F}$ an arbitrary HVB over $\mathbf{E}_q$. From [11, §3(c) p 57 Corollary of Theorem 9] (which actually holds over any compact Riemann surface), we know that $\mathscr{F}$ admits a tower of subbundles $0 = \mathscr{F}_0 \subset \cdots \subset \mathscr{F}_n = \mathscr{F}$ such that all the $\mathscr{F}_i/\mathscr{F}_{i-1}$, $i = 1, \ldots, n$, are line bundles.

Using the remarks on the matricial model in Subsubsection 1.1.1 and taking in account what was just said for line bundles, we see that $\mathscr{F} \simeq \mathscr{F}_B$, where:

$$B = \begin{pmatrix} u_1 & \dots & v_{1,n} \\ \vdots & \ddots & \vdots \\ 0 & \dots & u_n \end{pmatrix}, \quad \text{all } u_i = c_i x^{k_i}, c_i \in \mathbf{C}^*, k_i \in \mathbf{Z}, \quad \text{all } v_{i,j} \in \mathscr{O}(\mathbf{C}^*, 0).$$

We are going to exhibit a gauge matrix F such $F[A] = B$ in the form:

$$F = \begin{pmatrix} 1 & \dots & f_{1,n} \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}, \quad \text{all } f_{i,j} \in \mathscr{O}(\mathbf{C}^*, 0).$$

It follows that the matrix A has the same form as B, with the same diagonal coefficients u_i; but we want to (and will) obtain it with upper-diagonal coefficients $u_{i,j} \in \mathbf{C}(\{x\})$ instead of the $v_{i,j} \in \mathscr{O}(\mathbf{C}^*, 0)$. Putting relation $F[A] = B$ in equational form, we reduce it to:

$$\forall i, j \text{ s.t. } 1 \leq i < j \leq n \ , \ u_{i,j} + \sum_{i<l<j} (\sigma_q f_{i,l}) u_{l,j} + (\sigma_q f_{i,j}) u_j = v_{i,j} + \sum_{i<l<j} v_{i,l} f_{l,j} + u_i f_{i,j}.$$

The unknown are the $f_{i,j} \in \mathscr{O}(\mathbf{C}^*, 0)$ and the $u_{i,j} \in \mathbf{C}(\{x\})$. We solve those equations recursively for ascending values of $i - j$ (from 0 to n), in the form:

$$u_j \sigma_q f_{i,j} - u_i f_{i,j} = -u_{i,j} + \sum_{i<l<j} v_{i,l} f_{l,j} - \sum_{i<l<j} (\sigma_q f_{i,l}) u_{l,j} + v_{i,j}.$$

Each such equation is solved using Proposition 1.2; with the notation introduced at the end of Subsubsection 1.1.2, we set $(f_{i,j}, u_{i,j}) := (f, u_i u)$, where:

$$(f, u) := R_{c_j/c_i, k_j - k_i} \left(\frac{\sum_{i<l<j} v_{i,l} f_{l,j} - \sum_{i<l<j} (\sigma_q f_{i,l}) u_{l,j}}{u_i} \right).$$

□

The second statement will be greatly improved in Subsubsection 1.4.3 as a consequence of Proposition 1.15.

Caution 1.7. The proof actually shows more, *viz* that every HVB is isomorphic to some $\mathscr{F}_A$ with a *triangular* matrix $A \in \mathrm{GL}_n(\mathbf{C}(\{x\}))$. More precisely, if we start from an arbitrary q-difference module N and apply the corollary to $\mathscr{F}_N$, we see that there is a q-difference module $M = M_A$ with A triangular such that $\mathscr{F}_M \simeq \mathscr{F}_N$ *as HVBs*. However, this is an almost useless fact (at least for classification matters) since, the functor $M \rightsquigarrow \mathscr{F}_M$ not being full, it entails no structural relation between M and N. For a more useful statement, see the comment following Proposition 1.15.

1.2 Some particular classes of HVBs

In the (formal or analytic) classification of q-difference modules pure isoclinic ones play a central role. In the case of integral slopes, they have the form of a tensor product $M = M_1 \otimes M_2$, where M_1 has rank 1 and M_2 is flat. Therefore in that case $\mathscr{F}_M$ is the tensor product of a line bundle by a flat HVB. Hereunder, we provisionally call "pure" such HVBs (see Definition 1.8). In the general case of arbitrary slopes we shall see that the relevant notion is semi-stability.

1.2.1 Semi-stable and pure HVBs

Let $c \in \mathbf{C}^*$, $k \in \mathbf{Z}$. Then $\mathscr{F}_{cx^k}$ is a line bundle of degree (and slope) $-k$. Conversely, by Lemma 1.1, every line bundle can be described this way. Then k is unique (the opposite of the degree) and c is unique modulo $q^{\mathbf{Z}}$; argument: if the line bundle $\mathscr{F}$ has degree μ, then $\mathscr{F}_{x^{-\mu}} \otimes \mathscr{F}$ is a flat line bundle, whence it can be put in the form $\mathscr{F}_c$ with $c \in \mathbf{C}^*$ well determined modulo $q^{\mathbf{Z}}$. (So this yields a third proof for Lemma 1.1.)

Definition 1.8 (Provisional)**.** A *pure* HVB over $\mathbf{E}_q$ is the tensor product of a line bundle and a flat bundle.

Thus pure HVBs are those of the form $\mathscr{F}_{x^k A}$, where $k \in \mathbf{Z}$ is unique, and $A \in \mathrm{GL}_n(\mathbf{C})$, $\mathrm{Sp}A \subset \mathcal{C}_q$; such a A is unique up to conjugacy. In this Subsubsection, we want to find the corresponding notion when slopes are not supposed to be integral, more precisely, to characterize those HVBs which become pure after ramification (it is clear that pure HVBs stay pure after ramification, see Caution 1.9 herebelow).

So let $s \in \mathbf{N}^*$, $s \geq 2$ and $q' := q^{1/s}$ (naturally defined as $e^{2\mathrm{i}\pi\tau/s}$). The surjective morphism $x' \mapsto x := x'^s$ from $\mathbf{C}^*$ to itself sends $q'^{\mathbf{Z}}$ into $q^{\mathbf{Z}}$, the preimage of $q^{\mathbf{Z}}$ being $\mu_s q'^{\mathbf{Z}}$ (recall that μ_s is the group of s^{th} roots of unity in $\mathbf{C}^*$), so it induces a surjective morphism of groups $p : \mathbf{E}_{q'} \to \mathbf{E}_q$ with cyclic kernel $\mu_s q'^{\mathbf{Z}}/q'^{\mathbf{Z}} \simeq \mu_s$. Since it also is a holomorphic map of Riemann surfaces, it is actually an isogeny of elliptic curves. In particular, for every HVB $\mathscr{F}$ over $\mathbf{E}_q$, the pullback $p^*\mathscr{F}$ is an HVB over $\mathbf{E}_{q'}$. (In terms of q-difference modules, this corresponds to *ramification* $\mathbf{C}(\{x\}) \subset \mathbf{C}(\{x^{1/s}\})$ (see [16, 8.3.3]).

Caution 1.9. Under that ramification, $p^*\mathscr{F}_{x^k A} = \mathscr{F}_{x'^{ks} A}$, which confirms the previous statement about purity and ramification, but also shows the ambiguity of notation (displaying HVBs either over $\mathbf{E}_q$ or $\mathbf{E}_{q'}$).

We shall have use in Section 1.4 for the following result[3]:

Proposition 1.10. *Let $\mathscr{F}$ an HVB over $\mathbf{E}_q$ with slope in $\frac{1}{s}\mathbf{Z}$. Then $p^*\mathscr{F}$ has integral slope; and it is pure if, and only if, $\mathscr{F}$ is semi-stable.*

Proof. The HVBs $\mathscr{F}$ and $p^*\mathscr{F}$ have the same rank, but the degree of the latter is s times the degree of the former (see Appendix A.1), whence the first statement. The second statement is an immediate consequence of the combination of the two lemmas that follow. □

Before stating and proving the two necessary lemmas, we mention a particular case of the proposition.

Corollary 1.11. *The flat HVBs are exactly the semi-stable HVBs of degree (and slope)* 0.

Lemma 1.12. *The HVB $\mathscr{F} \to \mathbf{E}_q$ is pure if, and only if, it is a semi-stable bundle with integral slope.*

[3] I have been stuck a long time looking for such a characterisation, for which I thank Yan Soibelman.

Proof. Let $\mathscr{F}$ an arbitrary bundle and $\mathscr{L}$ a line bundle. Then $\mathscr{F}$ is pure (resp. semi-stable) if, and only if, $\mathscr{F} \otimes \mathscr{L}$ is; and since (see Appendix A.1):

$$\text{degree } \mathscr{F} \otimes \mathscr{L} = \text{degree } \mathscr{F} + (\text{rank } \mathscr{F})(\text{degree } \mathscr{L}),$$

the above statement boils down to: $\mathscr{F} \to \mathbf{E}_q$ is flat if, and only if, it is a semi-stable bundle with slope (or degree) 0.
So let first $\mathscr{F}$ be flat, say $\mathscr{F} = \mathscr{F}_A$, $A \in \mathrm{GL}_n(\mathbf{C})$. Using a trigonalisation of A we see that $\mathscr{F}$ comes from successive extensions of flat line bundles, which are semi-stable (as line bundles) and of degree 0 (being of the form $\mathscr{F}_c$, $c \in \mathbf{C}^*$). We now use the fact that, by [17, Première Partie, I B Corollaire 7 p 17], an extension of semi-stable HVBs having the same slope μ is a semi-stable HVB having slope μ.
Let conversely $\mathscr{F}$ be semi-stable with slope (or degree) 0. By [17, Première Partie, I B Theorème 10 p 18], $\mathscr{F}$ comes from successive extensions of stable line bundles of degree 0, so that, using the fact that (by Lemma A.1, see Subsection A.3 of Appendix A) an extension of flat bundles is flat, we may as well assume that $\mathscr{F}$ is stable of degree 0. But stable bundles are indecomposable (this is obvious from the definition), so we may appeal to Weil's Theorem [17, Première Partie, IV Proposition 36 p 40] which guarantees that an indecomposable HVB of degree 0 is flat. □

The next result is an extremely particular case of [12, Lemma 3.2.2], but in this simple case one can give a direct proof[4]:

Lemma 1.13. *Let $p : E' \to E$ an isogeny of elliptic curves and let $\mathscr{F}$ an HVB over E. Then $\mathscr{F}$ is semi-stable if, and only if, its pullback $\mathscr{F}' := p^*\mathscr{F}$ is.*

Proof. Assume $\mathscr{F}$ is not semi-stable and let $\mathscr{F}_1 \subset \mathscr{F}$ a destabilizing subbundle, *i.e.* $\mathscr{F}_1$ is neither $\mathscr{F}$ nor trivial and slope $\mathscr{F}_1 >$ slope $\mathscr{F}$. Then $p^*\mathscr{F}_1$ is a destabilizing subbundle of $p^*\mathscr{F}$ (it is neither $\mathscr{F}'$ nor trivial and pulling back p^* multiplies all slopes by the degree of the isogeny).
Assume conversely that $\mathscr{F}' := p^*\mathscr{F}$ is not semi-stable and let $\mathscr{F}'_1$ its maximal semi-stable subbundle [17, Première Partie, I A prop 2 p 15]. Every automorphism of the HVB $\mathscr{F}'$ preserves the properties characterizing uniquely $\mathscr{F}'_1$ (being neither $\mathscr{F}'$ nor trivial, inequalities on slopes and ranks), so $\mathscr{F}'_1$ is invariant by every automorphism of $\mathscr{F}'$, in particular by $\mathrm{Aut}(\mathscr{F}'/\mathscr{F})$. By Galois descent (see precisions further below), there is a unique subbundle $\mathscr{F}_1 \subset \mathscr{F}$ such that $\mathscr{F}'_1 = p^*\mathscr{F}_1$. Then clearly $\mathscr{F}_1$ is a destabilizing subbundle of $\mathscr{F}$ (same kind of argument as above). □

So we must now substantiate our appeal to Galois descent. We shall use the following theorem (in which X and Y denote schemes) that we quote completely from [6, Theorem 28]:
Let $p : X \to Y$ be a finite flat Galois morphism with Galois group G, and suppose that X and Y are both integral. Let $\mathscr{F}$ be a locally free sheaf on Y and suppose that $\mathcal{G}'$ is a coherent subsheaf of $p^*\mathscr{F}$ such that:
(i) For all $\sigma \in G$, $\sigma^*\mathcal{G}' = \mathcal{G}'$ as a subsheaf of $p^*\mathscr{F}$;
(ii) $p^*\mathscr{F}/\mathcal{G}'$ is torsion free.

[4] I heartily thank Arnaud Beauville for giving me the answer and reference and even explaining the main idea.

Then there exists a coherent subsheaf $\mathcal{G}$ of $\mathscr{F}$ such that $\mathcal{G}' = p^*\mathcal{G}$ as a subsheaf of $p^*\mathscr{F}$.

This of course applies to $p : E' \to E$. The proof of Lemma 1.13 leads to condition (i) as regards $\mathcal{G}' := \mathscr{F}'_1$, and condition (ii) comes from the fact that it is a subbundle (whence saturated as a subsheaf). Now we want $\mathscr{F}_1 := \mathcal{G}$ to be not only coherent but locally free: but this is automatic since E is a nonsingular curve (every stalk $\mathcal{G}_a$, $a \in E$, is a finitely generated submodule of the finitely generated free module $\mathscr{F}_a$ over the principal local ring $\mathscr{O}_{E,a}$).

Il doit y avoir moyen de faire plus élémentaire ?

1.2.2 Semi-stable HVBs have the form $\mathscr{F}_M$ with M pure isoclinic

We first recall some results of Atiyah [2] but in the forms given to them in [7] (which details the relationship of Atiyah's classification of HVBs over an elliptic curve with semi-stability).

1. (Lemma 1.1 of [7]): For every $n \in \mathbf{N}^*$, there is a unique indecomposable HVB I_n of rank n whose Jordan-Hölder tower has all quotients isomorphic to $\mathscr{O}_{\mathbf{E}_q}$. It admits a canonical filtration $F_0 \subset \cdots \subset F_n$ such that each $F_i \simeq I_i$ and each $F_i/F_{i-1} \simeq \mathscr{O}_{\mathbf{E}_q}$.

2. (Lemma 1.2 of [7]): Let $a \wedge n = 1$ and let λ a line bundle of degree a. There is a unique indecomposable HVB $W_n(a, \lambda)$ having rank n and determinant λ.

3. (Theorem 1.6 of [7]): Every semi-stable HVB of slope a/n with $a \wedge n = 1$ is isomorphic to a direct sum of HVBs of the form $I_d \otimes W_n(a, \lambda)$.

Before going to the main result of this Subsubsection, we state and prove a useful simple fact.

Lemma 1.14. *If M is pure isoclinic and indecomposable, then $\mathscr{F}_M$ is indecomposable.*

VÉRIFIER SI LA NOUVELLE FORMULATION (AVEC HYPOTHÈSE PURE ISOCLINIC) POSE PROBLÈME PAR LA SUITE.

Proof. Indeed a decomposition of $\mathscr{F}_M$ can be decribed by a projector, *i.e.* an idempotent endomorphism ϕ of $\mathscr{F}_M$. By "faithful fullness" of the functor $M \rightsquigarrow \mathscr{F}_M$ *when restricted to pairs of pure isoclinic q-difference modules having the same slope* (see the "caution" remark at the very end of Subsubsection 1.1.3), this comes from an idempotent endomorphism f of M. If M is indecomposable, f must be 0 or Id_M, so ϕ must be 0 or $\mathrm{Id}_{\mathscr{F}_M}$. □

Then we recall some facts from **CV4, compléter référence** about quasielementary q-difference modules. First, denoting K_m any irreducible Jordan block of size m, let $U_m := (K^m, \Phi_{K_m})$. (Up to isomorphism, this does not depend on the particular K_m selected.) Then U_m is indecomposable and its Jordan-Hölder tower has all quotients isomorphic to $\underline{\mathbf{1}} := (K, \sigma_q)$; the HVB associated to $\underline{\mathbf{1}}$ is $\mathscr{F}_1$, *i.e.* $\mathscr{O}_{\mathbf{E}_q}$. By lemma 1.14, $\mathscr{F}_{U_m}$ is indecomposable. Using the first fact recalled above from [7], we see that $\mathscr{F}_{U_m} \simeq I_n$.

Justifier quelque part l'indécomposabilité de U_m.

Next, let $L(r,d,a) := (K^r, \Phi_C)$, where:

$$C := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ (ax^d)^{-1} & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

From Proposition **ref ?? dans CV4**, if $a \wedge d = 1$ then $L(r,d,a)$ is indecomposable, so, again by the above lemma, $\mathscr{F}_C$ is indecomposable. It has rank r and determinant $\lambda := \mathscr{F}_{ax^d}$, so, by the second quoted fact from [7], it is $W_r(d,\lambda)$, where $\lambda := \mathscr{F}_{ax^d}$.

Proposition 1.15. *Every semi-stable HVB over $\mathbf{E}_q$ is isomorphic to $\mathscr{F}_P$ for some pure isoclinic q-difference module P.*

Proof. Given that the functor $M \rightsquigarrow \mathscr{F}_M$ is additive and $\otimes$-compatible, it is enough to prove the proposition for the HVBs I_n and $W_n(a,\lambda)$ mentioned above. We did it for I_n; and since every line bundle over $\mathbf{E}_q$ is isomorphic to a $\mathscr{F}_{ax^d}$, we also did it for every $W_r(d,\lambda)$. □

1.2.3 The formal classification is equivalent to the analytic classification of pure q-difference modules

It follows from **refs dans CV4 et MSM287** that the formal classification of analytic q-difference modules is equivalent to the analytic classification of pure q-difference modules. More precisely, *the functor $\mathcal{P}_q \to \hat{\mathscr{E}}_q$ induced by the formalisation functor $\mathscr{E}_q \to \hat{\mathscr{E}}_q$ is an equivalence of tensor categories.*

On the other hand, restricting the functor $M \rightsquigarrow \mathscr{F}_M$ to source $\mathcal{P}_q$ and corestricting it to target $\mathscr{V}_q$, we obtain an essentially surjective faithful $\otimes$-compatible functor. That functor is not full (usual counter-example: $\theta_q : \mathscr{F}_x \to \mathscr{F}_1$ is a morphism in the target not coming from the source). However, we have the following important result from [18]

Theorem 1.1. *That functor induces a bijection between the sets[5] of equivalence classes of both categories.*

Proof. This is immediate from, on the one hand, the formal classification of q-difference modules (due to van der Put and Reversat [18], see **ref à CV4 et MSM287**); and, on the other hand, Atiyah's classification of HVBs over $\mathbf{E}_q$ as recalled in Subsubsection 1.2.2. □

We can summarize that fact by saying that *the formal classification of (formal or analytic) q-difference modules is equivalent to Atiyah's classification.*

However, *this is not part of an equivalence of categories*, so we still have some work to do.

[5] Exercise: confirm that those are indeed sets.

1.3 Anti-HN filtrations

To combine the above results with those from [16, Chapter 10] on the canonical slope filtration of q-difference modules, we shall use some generalities on filtered objects in abelian categories (here: the category $\mathscr{E}_q$ of q-difference modules over $\mathbf{C}(\{x\})$ and the category $\mathscr{S}_q$ of coherent sheaves over $\mathbf{E}_q$) and their behaviour under the effect of an exact functor (here: $M \rightsquigarrow \mathscr{F}_M$). Those generalities are summarized in Appendix B.

1.3.1 The image of the slope filtration by the functor $M \rightsquigarrow \mathscr{F}_M$

Recall that K denotes $\mathbf{C}(\{x\})$. To begin with, let P a pure isoclinic q-difference module of rank n and slope μ. If μ is integral, $P \simeq (K^n, \Phi_{x^\mu A})$ for some $A \in \mathrm{GL}_n(\mathbf{C})$, so $\mathscr{F}_M \simeq \mathscr{F}_{x^\mu A} \simeq \mathscr{F}_{x^\mu} \otimes \mathscr{F}_A$ is a pure HVB of integral slope $\lambda := -\mu$, whence a semi-stable bundle by Subsubsection 1.2.1.

If μ is not integral, let $\mu = d/r$ where r can be taken such that $r|n$. Set $q' := q^{1/r}$, $x' := x^{1/r}$ and $K' := K[x']$. Therefore $P' := K' \otimes_K P$ is a pure isoclinic q'-difference module of integral slope $\mu' = r\mu = d$. Writing $p : \mathbf{E}_{q'} \to \mathbf{E}_q$ the natural isogeny, we have $\mathscr{F}_{P'} \simeq p^* \mathscr{F}_P$, so, again by Subsubsection 1.2.1, $\mathscr{F}_P$ is a semi-stable HVB with non integral slope $\lambda := -\mu$.

Now let M a q-difference module over K and let $\mu_1 > \cdots > \mu_k$ its slopes. Using the canonical slope filtration:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M, \text{ each } P_i := M_i/M_{i-1} \text{ pure isoclinic of slope } \mu_i,$$

and the functor $M \rightsquigarrow \mathscr{F}_M$ from Proposition 1.6, we obtain a filtered HVB:

$$0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_k = \mathscr{F}_M$$

such that each $\mathscr{F}_i = \mathscr{F}_{M_i}$, $i = 1, \ldots, k$, is a subbundle and each $\mathscr{F}_i/\mathscr{F}_{i-1}$ is (isomorphic to) the HVB $\mathscr{F}_{P_i}$, which is a semi-stable HVB of slope $\lambda_i := \mu_i$. Note that *the slopes of the quotient bundles are increasing*: $\lambda_1 < \cdots < \lambda_k$.

Remark 1.16. It is not very clear how the above filtration relates to the plain Harder-Narasimhan filtration on $\mathscr{F}_M$. If for instance we start from $A := \begin{pmatrix} x & 1 \\ 0 & 1 \end{pmatrix}$, then we see from [16, Subsection 10.4.2 p.354] that $\mathscr{F}_A$ is stable, while its inherited filtration is $0 \subset \mathscr{F}_x \subset \mathscr{F}_A$ (thus with second quotient $\mathscr{F}_1$).

1.3.2 Anti-HN filtration on a vector bundle

We are led to axiomatize as follows the properties of the filtration on $\mathscr{F}_M$ induced by the canonical slope filtration on M.

Definition 1.17. An *anti-HN filtration*[6] of an HVB $\mathscr{F}$ over $\mathbf{E}_q$ is a finite filtration by subbundles such that the successive quotients are semi-stable HVBs with *increasing* slopes.

We now fix the notations and improve the terminology. In its concrete form, the anti-HN filtration of $\mathscr{F}$ is a *tower* (or *flag*) $0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_k = \mathscr{F}$

[6] The terminology appears in talks by Kontsevich and Soibelman on (a very general form of) the Riemann-Hilbert correspondence; they deal with coherent sheaves, not only HVBs.

of subbundles such that each $\mathscr{F}_i/\mathscr{F}_{i-1}$ is a semi-stable HVB and the slopes $\lambda_i :=$ slope $\mathscr{F}_i/\mathscr{F}_{i-1}$ are increasing: $\lambda_1 < \cdots < \lambda_k$.

Alternatively (and more properly), this can be described as an *ascending* $\mathbf{Q}$-*filtration* by subbundles $\mathscr{F}_{\leq\lambda}$, $\lambda \in \mathbf{Q}$. The filtration is exhaustive (*i.e.* $\bigcup \mathscr{F}_{\leq\lambda} = \mathscr{F}$) and separated (*i.e.* $\bigcap \mathscr{F}_{\leq\lambda} = 0$) and therefore it has a finite number of *breaks* or *jumps*, *i.e.* values af the index λ such that the inclusion:

$$\mathscr{F}_{<\lambda} := \bigcup_{\lambda'<\lambda} \mathscr{F}_{\leq\lambda'} \subset \mathscr{F}_{\leq\lambda}$$

is strict. Those breaks are the λ_i above.

Then the defining condition (for being an anti-HN filtered HVB) is that every non trivial $\mathscr{F}_{(\lambda)} := \mathscr{F}_{\leq\lambda}/\mathscr{F}_{<\lambda}$ (*i.e.* those are jumps) is a semi-stable bundle of slope λ. (It is the fact that the non trivial $\mathscr{F}_{(\lambda)}$ are vector bundles which guarantees that the number of jumps is finite.)

Now since we shall be producing such filtrations by applying the functor $M \rightsquigarrow \mathscr{F}_M$ to the *descending* slope filtrations of q-difference modules, it seems more comfortable to rather consider our anti-HN filtrations as defined by *descending* $\mathbf{Q}$-filtrations $\mathscr{F}^{\geq\mu}$, $\mu \in \mathbf{Q}$, so we set $\mathscr{F}^{\geq\mu} := \mathscr{F}_{\leq-\mu}$, and the above rules become that at the breaks $\mu_i = -\lambda_i$ we find semi-stable bundles:

$$\mathscr{F}^{(\mu_i)} := \mathscr{F}^{\geq\mu_i}/\mathscr{F}^{>\mu_i} = \mathscr{F}_{(\lambda_i)} \text{ of slopes } \lambda_i = -\mu_i \text{ such that } \mu_1 > \cdots > \mu_k.$$

1.3.3 Morphisms

Let $f : M \to N$ a morphism of q-difference modules, whence (by Proposition 1.6) a morphism $\phi : \mathscr{F}_M \to \mathscr{F}_N$. From the functorial properties of the canonical slope filtration [16, Cor. 10.30 p. 336], we know that:

$$\forall \mu \in \mathbf{Q}\ ,\ f(M^{\geq\mu}) = f(M) \cap N^{\geq\mu}.$$

In terms of the slope filtrations on M and N, this means that the morphism f is strict (Appendix B). In view of the results of Subsection B.3, in particular Theorem B.1, it follows that the morphism of filtered HVBs $\mathscr{F}_M \to \mathscr{F}_N$ (of course, endowed with the anti-HN filtrations) is strict. This uses the exactness of the functor $M \rightsquigarrow \mathscr{F}_M$, ensured by Proposition 1.6.

Since vector bundles are not an abelian category and since strict morphisms are not the morphisms of a category (see Example B.2 in Appendix B), we must first give some precisions on the categories and functors at stake.

1.3.4 The functor

Recall $\mathscr{E}_q$ the category of all q-difference modules over $\mathbf{C}(\{x\})$ and $\mathscr{S}_q$ the category of coherent sheaves over $\mathbf{E}_q$. Their respective categories of $\mathbf{Q}$-filtered objects (Appendix B) are $\mathrm{Fil}_{\mathbf{Q}}(\mathscr{E}_q)$ and $\mathrm{Fil}_{\mathbf{Q}}(\mathscr{S}_q)$. We have *a priori* two functors:

- A functor $M \rightsquigarrow \underline{M}$ from $\mathscr{E}_q$ to $\mathrm{Fil}_{\mathbf{Q}}(\mathscr{E}_q)$ which associates to the q-difference module M itself endowed with the Adams filtration;
- A functor $\underline{M} \rightsquigarrow \underline{\mathscr{F}}_M$ from $\mathrm{Fil}_{\mathbf{Q}}(\mathscr{E}_q)$ to $\mathrm{Fil}_{\mathbf{Q}}(\mathscr{S}_q)$ which is the natural extension of $M \rightsquigarrow \mathscr{F}_M$ to filtered objects (Subsection B.3). This one exists because all morphisms in $\mathscr{E}_q$ are compatible with the slope filtration.

Composing them, we get the functor $M \rightsquigarrow \underline{\mathscr{F}}_M$ from $\mathscr{E}_q$ to $\mathrm{Fil}_{\mathbf{Q}}(\mathscr{S}_q)$ which interests us. We know *a priori* the following facts about that functor:

- All image objects in $\mathrm{Fil}_{\mathbf{Q}}(\mathscr{S}_q)$ are actually HVBs endowed with anti-HN filtrations (Subsubsection 1.3.1);
- All image morphisms are strict (Theorem B.1).

1.4 Local q-RHB

1.4.1 The main result

Recall that $\mathscr{V}_q$ is the full subcategory of $\mathscr{S}_q$ having as objects the HVBs; therefore $\mathrm{Fil}_{\mathbf{Q}}(\mathscr{V}_q)$ is a full subcategory of $\mathrm{Fil}_{\mathbf{Q}}(\mathscr{S}_q)$. We denote $\mathrm{Fil}^*_{\mathbf{Q}}(\mathscr{V}_q)$ its full subcategory having as objects HVBs endowed with an anti-HN filtration.

Discussion rapide sur le foncteur $M \rightsquigarrow \underline{\mathscr{F}}_M$: il est exact (ça utilise le fait que les morphismes sont stricts), C-lineaire, fidèle, $\otimes$-compatible; pour ce dernier point, voir Subsubsection B.1.3 dans l'appendice.

Theorem 1.2. *The functor $M \rightsquigarrow \underline{\mathscr{F}}_M$ from $\mathscr{E}_q$ to $Fil_{\mathbf{Q}}(\mathscr{S}_q)$ is exact, $\mathbf{C}$-linear, fully faithful and $\otimes$-compatible. Its essential image is $Fil^*_{\mathbf{Q}}(\mathscr{V}_q)$.*

Proof. The only statements not yet proven are those about being full and the essential image. They are respectively tackled in Subsubsections 1.4.2 and 1.4.3. □

It readily follows from the theorem that all morphisms in $\mathrm{Fil}^*_{\mathbf{Q}}(\mathscr{V}_q)$ are strict.

Let us explain the underlying idea of both proofs of the stated facts. Using pullbacks by the etale covering $\mathbf{C}^* \to \mathbf{E}_q$ we know from Appendix A.2 that every HVB over $\mathbf{E}_q$ has the form $\mathscr{F}_A$, $A \in \mathrm{GL}_n(\mathscr{O}(\mathbf{C}^*))$; and that every morphism $\mathscr{F}_A \to \mathscr{F}_B$ comes from a (rectangular) matrix F over $\mathscr{O}(\mathbf{C}^*)$ such that $(\sigma_q F)A = BF$.

The essential image statement in the theorem then says that A can be taken in $\mathrm{GL}_n(\mathbf{C}(\{x\}))$; and the fullness statement that, if A and B are over $\mathbf{C}(\{x\})$, then so is F. So we see them as *regularization* properties, saying that a matrix over $\mathscr{O}(\mathbf{C}^*)$ can be replaced by one over $\mathbf{C}(\{x\})$. However, the procedure allowing to do so mixes $\mathscr{O}(\mathbf{C}^*)$ and $\mathbf{C}(\{x\})$, so it is actually better to start from matrices over $\mathscr{O}(\mathbf{C}^*, 0)$.

Such regularization procedures for matrices over $\mathscr{O}(\mathbf{C}^*, 0)$ have been emphasized in [15] in relation with so-called "homological equation". They will appear here in the form of lemmas 1.18 and 1.20.

Thus, for the characterisation of the essential image, those typical q-difference techniques (to be used in Subsubsections 1.4.2 and 1.4.3) come in complement of vector bundles techniques (around Atiyah's classification) previously used in Subsubsection 1.2.2.

1.4.2 The functor $M \rightsquigarrow \underline{\mathscr{F}}_M$ is full

The following lemma is really the clue of the interest of anti-HN filtrations for classification. It will help prove fullness in the proposition following it.

Lemma 1.18. *(i) Let $B_i \in GL_{r_i}(\mathbf{C}(\{x\}))$, $i = 1, 2$, be pure isoclinic of slopes μ_i such that $\mu_1 > \mu_2$. Let $X \in Mat_{r_1,r_2}(\mathscr{O}(\mathbf{C}^*, 0))$ and $Y \in Mat_{r_1,r_2}(\mathbf{C}(\{x\}))$ be such that:*

$$(\sigma_q X)B_2 - B_1 X = Y.$$

Then $X \in Mat_{r_1,r_2}(\mathbf{C}(\{x\}))$.
(ii) Let $B_i \in GL_{r_i}(\mathbf{C}(\{x\}))$, $i = 1, 2$, be pure isoclinic with the same slope μ and let $X \in Mat_{r_1,r_2}(\mathscr{O}(\mathbf{C}^*, 0))$ be such that:

$$(\sigma_q X)B_2 = B_1 X.$$

Then $X \in Mat_{r_1,r_2}(\mathbf{C}(\{x\}))$.

Proof. (i) This is proved as Lemma 15.64 of [16, 15.3.1.1] under the assumption that $\mu_1, \mu_2 \in \mathbf{Z}$ and the B_i are in "flat form" $B_i = x^{\mu_i} A_i$, $A_i \in \mathrm{GL}_{r_i}(\mathbf{C})$. We now relax this assumption in two steps.
If $\mu_1, \mu_2 \in \mathbf{Z}$ but the B_i are not (yet) in "flat form", we take $F_i \in \mathrm{GL}_{r_i}(\mathbf{C}(\{x\}))$ and $C_i \in \mathrm{GL}_{r_i}(\mathbf{C})$, $i = 1, 2$, such that $B_i = F_i[B_i']$, where $B_i' := x^{\mu_i} C_i$. Then, setting $X' := F_1^{-1} X F_2$ and $Y' := (\sigma_q F_1)^{-1} Y F_2$, we see that:

$$(\sigma_q X)B_2 - B_1 X = Y \iff (\sigma_q X')B_2' - B_1' X' = Y'.$$

Since $X' \in \mathrm{Mat}_{r_1,r_2}(\mathbf{C}(\{x\}))$, resp. $X' \in \mathrm{Mat}_{r_1,r_2}(\mathscr{O}(\mathbf{C}^*, 0))$, is equivalent to $X \in \mathrm{Mat}_{r_1,r_2}(\mathbf{C}(\{x\}))$, resp. $X \in \mathrm{Mat}_{r_1,r_2}(\mathscr{O}(\mathbf{C}^*, 0))$, and the same for Y', Y, the result for those B_i is reduced to the same for the B_i', *i.e.* to Lemma 15.64 mentioned above.
Now let the μ_i be arbitrary and let $r \in \mathbf{N}^*$ a common denominator. Ramifying with $x' := x^{1/r}$ (and $q' := q^{1/r}$) entails that $X \in \mathrm{Mat}_{r_1,r_2}(\mathbf{C}(\{x'\})$. Since $\mathbf{C}(\{x'\}) \cap \mathscr{O}(\mathbf{C}^*, 0) = \mathbf{C}(\{x\})$, the conclusion flows.
(ii) The same reduction steps allow us to assume from start that $B_i = x^{\mu} A_i$, $A_i \in \mathrm{GL}_{r_i}(\mathbf{C})$ and then, writing as a Laurent power series $\sum_{d \in \mathbf{Z}} x^d X_d$:

$$(\sigma_q X)B_2 = B_1 X \iff \forall d \in \mathbf{Z}\ ,\ q^d X_d B_2 = B_1 X_d.$$

The right hand side equation has a non zero solution only if $q^d B_2$ and B_1 have intersecting spectra, *i.e.* for a finite set of values of d; so in this particular case X is (Laurent) polynomial. □

Proposition 1.19. *Let M, N two q-difference modules and $F : \mathscr{F}_M \to \mathscr{F}_N$ a morphism of HVBs compatible with their anti-HN filtrations. Then F "is" actually a morphism of q-difference modules from M to N.*

Proof. We take our q-difference modules in the form $M = (K^m, \Phi_A)$ and $N = (K^n, \Phi_B)$, so that (by Subsubsection 1.1.1) $\mathscr{F}_M = \mathscr{F}_A$ and $\mathscr{F}_N = \mathscr{F}_B$; and a morphism $\mathscr{F}_M \to \mathscr{F}_N$, resp. $M \to N$, is a matrix F such that $(\sigma_q F)A = BF$ having coefficients in $\mathscr{O}(\mathbf{C}^*, 0)$, resp. in $\mathbf{C}(\{x\})$. So what we want to show is that the $F : \mathscr{F}_M \to \mathscr{F}_N$ mentioned in the statement of the proposition really has coefficients in $\mathbf{C}(\{x\})$.
We assume A and B in block-triangular forms reflecting their respective filtration: we let $\mu_1 > \cdots > \mu_k$ the totalities of all slopes of A and B; we call $r_1, \ldots, r_k \in \mathbf{N}$, resp. $s_1, \ldots, s_k \in \mathbf{N}$, their multiplicities as slopes of A, resp. of B. The respective diagonals of blocks are $\mathrm{Diag}(A_1, \ldots, A_k)$ (each

$A_i \in \mathrm{GL}_{r_i}(\mathbf{C}(\{x\}))$) and $\mathrm{Diag}(B_1, \dots, B_k)$ (each $B_i \in \mathrm{GL}_{s_i}(\mathbf{C}(\{x\}))$)and the rectangular upper blocks of A, resp. B are the $U_{i,j} \in \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{x\}))$, resp. the $V_{i,j} \in \mathrm{Mat}_{s_i,s_j}(\mathbf{C}(\{x\}))$.

Traitement des blocs vides ?

The fact that F respects the filtrations of $\mathscr{F}_A$ and F_B induced by those of A,B means that it consists of rectangular blocks $F_{i,j}$ that are non zero only for $i \leq j$. The relation $(\sigma_q F)A = BF$ implies first that $(\sigma_q F_{i,i})A_i = B_i F_{i,i}$ for $i = 1, \dots, k$. By (ii) of Lemma 1.18, this implies that all $F_{i,i}$ have coefficients in $\mathbf{C}(\{x\})$.

Now for blocks (i, j) such that $i < j$ the relation $(\sigma_q F)A = BF$ writes:

$$(\sigma_q F_{i,j})A_j - B_i F_{i,j} = V_{i,j}F_{j,j} - (\sigma_q F_{i,i})U_{i,j} + \sum_{i<l<j} V_{i,l}F_{l,j} - \sum_{i<l<j} (\sigma_q F_{i,l})U_{l,j}.$$

By induction on $j - i$ (since all $l - i$ and $j - l$ are $< j - i$), we can apply (i) of Lemma 1.18 and conclude that all $F_{i,j}$ have coefficients in $\mathbf{C}(\{x\})$. □

1.4.3 The essential image of the functor $M \rightsquigarrow \underline{\mathscr{F}}_M$

Recall that, by Proposition 1.15 of Subsubsection 1.2.2, every semi-stable HVB over $\mathbf{E}_q$ is isomorphic to $\mathscr{F}_P$ for some pure isoclinic q-difference module P over $\mathbf{C}(\{x\})$. The clue to the dévissage is the following regularization lemma.

Lemma 1.20. *Let A, B pure isoclinic matrices over $\mathbf{C}(\{x\})$ of respective slopes $\mu > \nu$ and ranks r, s. Let $W \in Mat_{r,s}(\mathscr{O}(\mathbf{C}^*, 0))$. Then there exist $F \in Mat_{r,s}(\mathscr{O}(\mathbf{C}^*, 0))$ and $V \in Mat_{r,s}(\mathbf{C}(\{x\}))$ such that $V + (\sigma_q F)B = AF + W$.*

Proof. As in Lemma 1.18, ramification and gauge equivalence allow us to assume that A and B are in "flat form". But then it is (up to notation and the convention on $|q|$) the content of [15, Lemma 2.9]. □

Let us denote temporarily $Red(A, B, W)$ the pair (F, V).

Lemma 1.21. *Let $A \in GL_n(\mathbf{C}(\{x\}))$ with slopes $\mu_1 > \cdots > \mu_k$ having multiplicities $r_1, \dots, r_k$ and let $B \in GL_s(\mathbf{C}(\{x\}))$ pure isoclinic with slope $\nu < \mu_k$. Let $\mathscr{F}$ a HVB such that there is an exact sequence:*

$$0 \to \mathscr{F}_A \to \mathscr{F} \to \mathscr{F}_B \to 0.$$

Then $\mathscr{F}$ is isomorphic to $\mathscr{F}_C$ for some $C \in GL_{n+s}(\mathbf{C}(\{x\}))$.

En fait on prouve plus et ça doit pouvoir se formuler en termes de surjectivité de $\mathbf{Ext}(B, A) \to \mathbf{Ext}(\mathscr{F}_B, \mathscr{F}_A)$.

Proof. We let the reader check that we may assume that A is in its usual block-triangular form (as in the proof of Proposition 1.19). Then $\mathscr{F}$ is isomorphic to $\mathscr{F}_D$ for some block-triangular matrix:

$$D = \begin{pmatrix} A & W \\ 0 & B \end{pmatrix} \in \mathrm{GL}_{n+s}(\mathscr{O}(\mathbf{C}^*, 0)).$$

Note that W is a column of blocks $W_i \in \mathrm{Mat}_{r_i,s}(\mathscr{O}(\mathbf{C}^*, 0))$, $i = 1, \dots, k$.

We are going to replace D by $C = \begin{pmatrix} A & V \\ 0 & B \end{pmatrix}$, where V is a column of blocks $V_i \in$

$\mathrm{Mat}_{r_i,s}(\mathbf{C}(\{x\}))$. We shall do so through a gauge transformation $\begin{pmatrix} I_n & F \\ 0 & I_s \end{pmatrix}$, where F is a column of blocks $F_i \in \mathrm{Mat}_{r_i,s}(\mathscr{O}(\mathbf{C}^*, 0))$. Equality $F[C] = D$ will ensure that $\mathscr{F} \simeq \mathscr{F}_C = \mathscr{F}_{M_C}$ as wanted. Now:

$$F[C] = D \iff \forall i = 1, \dots, k \ , \ V_i + (\sigma_q F_i)B = W + A_i F_i + \sum_{i<j\leq k} U_{i,j} F_j.$$

We consider those k equalities as equations with unknowns F, V and solve them inductively from $i = k$ to $i = 1$ using Lemma 1.20:

$$(F_i, V_i) := Red\left(A_i, B, W + \sum_{i<j\leq k} U_{i,j} F_j\right).$$

□

Proposition 1.22. *Every HVB endowed with an anti-HN filtration is a $\underline{\mathscr{F}}_M$.*

Proof. Given the fact that every semi-stable HVB has the form $\mathscr{F}_P$ for some pure isoclinic q-difference module P (Subsubsection 1.2.2) and the above dévissage lemma, this is immediate by induction on the number of steps in the filtration. (One must check of course that the slopes behave as they should, this again is left to the reader.) □

Part I
Appendices

A Holomorphic vector bundles over an elliptic curve

Recall that we abbreviate HVB(s) for holomorphic vector bundle(s) over a Riemann surface, in particular over the elliptic curve $\mathbf{E}_q$ (thus a *compact* Riemann surface) or over its covering $\mathbf{C}^*$ (thus an *open* Riemann surface). General references are [11], also see [16, Appendices B.5.5,B.5.6]; and, for more specialized results, [2, 6, 7, 10, 12, 13, 17].

A.1 Generalities

From the general theory, we shall retain that every HVB $\mathscr{F}$ over a Riemann surface X has a rank $\operatorname{rank}\mathscr{F} = r \in \mathbf{N}$ and an associated locally free $\mathscr{O}_X$-module of rank r (the sheaf of its holomorphic sections). It also has a determinant $\det\mathscr{F}$ which is a line bundle.

If X is compact, every HVB also has a degree $\operatorname{degree}\mathscr{F} = d \in \mathbf{Z}$, whence a slope $\operatorname{slope}\mathscr{F} = d/r \in \mathbf{Q}$. The degree of a line bundle is the degree of the divisor of any of its meromorphic sections and the degree of a general HVB is the degree of its determinant.

Let $\mathscr{F}, \mathscr{F}'$ arbitrary HVBs and $\mathscr{L}, \mathscr{L}'$ line bundles. Then:

$$\operatorname{rank}(\mathscr{F}\otimes\mathscr{F}') = (\operatorname{rank}\mathscr{F})(\operatorname{rank}\mathscr{F}'),\ \operatorname{degree}(\mathscr{L}\otimes\mathscr{L}') = \operatorname{degree}\mathscr{L} + \operatorname{degree}\mathscr{L}',$$
$$\det(\mathscr{F}\otimes\mathscr{F}') = (\det\mathscr{F})^{\otimes(\operatorname{rank}\mathscr{F}')}\otimes(\det\mathscr{F}')^{\otimes(\operatorname{rank}\mathscr{F})},$$
$$\text{whence } \operatorname{slope}(\mathscr{F}\otimes\mathscr{F}') = \operatorname{slope}\mathscr{F} + \operatorname{slope}\mathscr{F}'.$$

Also, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of HVBs, then:

$$\operatorname{rank}\mathscr{F} = \operatorname{rank}\mathscr{F}' + \operatorname{rank}\mathscr{F}'',\ \operatorname{degree}\mathscr{F} = \operatorname{degree}\mathscr{F}' + \operatorname{degree}\mathscr{F}'',$$
$$\text{whence } \operatorname{slope}\mathscr{F} \in [\operatorname{slope}\mathscr{F}', \operatorname{slope}\mathscr{F}'']\,.$$

Over an open Riemann surface, every HVB is trivial. Over a compact Riemann surface, every HVB is meromorphically trivial.

Il faut parler du degré, du rang et de la pente de $p^*\mathscr{F}$ si p est une isogénie (ou un revêtement non ramifié).

A.2 Matricial models

A general construction of HVBs over a Riemann surface X as deduced from trivial HVBs over the universal covering $\tilde{X}$ of X, equivariant under the action of the fundamental group $\operatorname{Aut}(\tilde{X}/X)$, is described in [10, 17]. It rests on the fact that a vector bundle over a simply connected surface is trivial.

Instead of the universal covering of $\mathbf{E}_q$, we shall take advantage of the fact that the source $\mathbf{C}^*$ of the etale covering $\pi : \mathbf{C}^* \to \mathbf{E}_q$ is an open Riemann surface. Therefore, for every rank n HVB $\mathscr{F}$ over $\mathbf{E}_q$, the pullback $\pi^*\mathscr{F}$ is a trivial HVB $\mathbf{C}^*\times\mathbf{C}^n$ endowed with an equivariant action of the group $\operatorname{Aut}(\mathbf{C}^*/\mathbf{E}_q)$ of the

covering. The latter group may be realized as the cyclic group $q^{\mathbf{Z}}$ and its action is therefore determined by the action of its generator q:

$$q.(x,X) = (qx, A(x)X), \text{ where } A \in \mathrm{GL}_n(\mathscr{O}(\mathbf{C}^*)).$$

From this we obtain the geometric description of $\mathscr{F}$ as quotient of $\pi^*\mathscr{F}$ by that action:

$$\mathscr{F} \simeq \mathscr{F}_A := \frac{\mathbf{C}^* \times \mathbf{C}^n}{(x,X) \sim (qx, A(x)X)}.$$

The description of $\mathscr{F}$ as a locally free sheaf follows at once; for every open subset V of $\mathbf{E}_q$:

$$\mathscr{F}_A(V) = \{X \in \mathscr{O}(\pi^{-1}(V)) \mid \sigma_q X = AX\}.$$

If $A \in \mathrm{GL}_n(\mathscr{O}(\mathbf{C}^*))$ and $B \in \mathrm{GL}_p(\mathscr{O}(\mathbf{C}^*))$, one draws that the morphisms $\mathscr{F}_A \to \mathscr{F}_B$ are encoded by matrices $F \in \mathrm{Mat}_{p,n}(\mathscr{O}(\mathbf{C}^*))$ such that $(\sigma_q F)A = BF$. For any such matrix the maps $X \mapsto FX$ send each $\mathscr{F}_A(V)$ to $\mathscr{F}_B(V)$ and patch into a morphism of sheaves $\mathscr{F}_A \to \mathscr{F}_B$.

A.3 Flat HVBs over $\mathbf{E}_q$

In the case of flat bundles, the general construction alluded to at the beginning of Subsection A.2 can be expressed in terms of linear representations of the fundamental group $\pi_1(\mathbf{E}_q)$ [17]. The corresponding notion in our case would involve linear representations of $\mathrm{Aut}(\mathbf{C}^*/\mathbf{E}_q)$. That group being cyclic, this amounts to matrices in $\mathrm{GL}_n(\mathbf{C})$. We now detail that fact.

We call here *flat* an HVB which (in the sense of differential geometry) *could* be endowed with a flat connection. Here "flat" is therefore taken as a property, not an additional structure. As references for flat HVBs, see [16, 13.2.6 and B.5.5] and [11, §6, p 96] (also see [17, Première Partie, IV]).

The general theory ensures that flat HVBs have degree 0 (whence slope 0) and that a direct sum of indecomposable HVBs is flat if, and only if, each summand has degree 0. Let $A \in \mathrm{GL}_n(\mathbf{C})$. Then $\mathscr{F}_A$ is a flat vector bundle.

Conversely, every flat HVB can be described this way. This results from the characterization of flat bundles in [11] as those for which a defining cocycle may be found with values in $\mathrm{GL}_n(\mathbf{C})$. (Actually, Gunning takes it as a definition.)

Since A may be taken up to gauge transformation, one can moreover require that $\mathrm{Sp}A \subset \mathcal{C}_q$; and A is then unique up to conjugacy.

Lemma A.1. *An extension of flat HVBs is flat.*

Proof. Let $A \in \mathrm{GL}_n(\mathbf{C})$ and $B \in \mathrm{GL}_m(\mathbf{C})$ with spectra included in $\mathcal{C}_q$. Appealing again to [16] (also see [11, §4(a), p 61]), we see that an extension of $\mathscr{F}_B$ by $\mathscr{F}_A$ has the form $\mathscr{F}_C$, where:

$$C = \begin{pmatrix} A & U \\ 0 & B \end{pmatrix}, \quad U = \sum x^k U_k \in \mathrm{Mat}_{n,m}(\mathscr{O}(\mathbf{C}^*, 0)); \text{ we let } C_0 = \begin{pmatrix} A & U_0 \\ 0 & B \end{pmatrix}.$$

We shall produce an isomorphism $\mathscr{F}_{C_0} \to \mathscr{F}_C$ in the form $\begin{pmatrix} I_n & F \\ 0 & I_m \end{pmatrix}$. This amounts to solving $(\sigma_q F)B - AF = U - U_0$, equivalently $F = \sum x^k F_k$, with (convergence matters *i.e.* $F \in \mathrm{Mat}_{n,m}(\mathscr{O}(\mathbf{C}^*))$ temporarily left aside):

$$F_0 = 0 \text{ and } \forall k \in \mathbf{Z} \setminus \{0\}\ ,\ q^k F_k B - AF_k = U_k,\ \textit{i.e.}\ \Phi_{A,q^k B}(F_k) = -U_k.$$

Here we have introduced, for every $P \in \mathrm{GL}_n(\mathbf{C})$, $Q \in \mathrm{GL}_m(\mathbf{C})$, the endomorphism $\Phi_{P,Q} : M \mapsto PM - MQ$ of $\mathrm{Mat}_{m,n}(\mathbf{C})$. It is well known that $\Phi_{P,Q}$ is an isomorphism if, and only if, P and Q have disjoint spectra, so (due to the assumption on the spectra of A and B), all the above individual equations have unique solutions $F_k = -\Phi^{-1}_{A,q^k B}(U_k)$.
As for convergence, note that $\Phi_{A,q^k B}$ is equivalent to left multiplication by A, resp. by $-q^k B$, when $k \to +\infty$, resp. $k \to -\infty$, so that for any subordinate matricial norm, $\left\|\Phi^{-1}_{A,q^k B}\right\| \sim \left\|(q^k B)^{-1}\right\|$ when $k \to -\infty$, while $\left\|\Phi^{-1}_{A,q^k B}\right\| \sim \left\|A^{-1}\right\|$ when $k \to +\infty$. The first relation implies the convergence of $\sum_{k<0} F_k x^k$ on $\mathbf{C}^*$; and the second relation, that of $\sum_{k>0} F_k x^k$ on some $\overset{\circ}{\mathrm{D}}(0,R)$, $R > 0$. Therefore F converges on $\overset{\bullet}{\mathrm{D}}(0,R)$. □

B Filtered objects in an abelian category

General references for this Appendix are [3, 9, 8, 1], along with the first section of [5] for some facts on filtered objects. Also for the relationship between the various ways of dealing with descending filtrations (hereunder Subsubsection B.1.1) see [16, 10.2.2]. One should also mention, although I do not know if that can be considered as a legitimate reference, the Stacks project (Section 12.19, Filtrations), see the URL `https://stacks.math.columbia.edu/tag/0120`.

Letters $\mathscr{A}, \mathscr{B}$ denote abelian categories (see in particular [9, Chapitre I] and [8, Chapter II]). We use set-theoretic (better said: module-theoretic) notations and terminology for subobjects, quotient objects, etc, when it seems that there is no risk of illogism: for instance a subobject, resp. a quotient object of X may be denoted as $X' \subset X$, resp. X/X'. For a justification of that attitude, see [14, Chapter VIII] and the exercises at the end of [8, Chapter II §5] (which is, at any rate, a good introduction to abelian categories).

B.1 Filtered objects

B.1.1 Descending Q-filtrations

A *descending* $\mathbf{Q}$*-filtration* on an object X of the abelian category $\mathscr{A}$ is a decreasing family of subobjects $F^{\geq\mu}X$, $\mu \in \mathbf{Q}$:

$$\forall \mu, \nu \in \mathbf{Q}\ ,\ \mu \leq \nu \Longrightarrow F^{\geq\nu}X \subset F^{\geq\mu}X \subset X.$$

We shall only consider filtrations that are *exhaustive, separated and finite*. Without defining independently those three qualifications, let us describe equivalent data. Assuming $X \neq 0$, there is a tower of *distinct* subobjects:

$$0 = X_0 \subset \cdots \subset X_k = X \quad \text{(inclusions here are strict)}$$

and corresponding indexes (the *breaks*) $\mu_1 > \cdots > \mu_k$ in $\mathbf{Q}$ such that:

$$\forall \mu \in \mathbf{Q}\ ,\ \forall i \in \{0, \ldots, k\}\ ,\ F^{\geq\mu}X = X_i \Longleftrightarrow \mu_{i+1} < \mu \leq \mu_i.$$

(It is understood that $\mu_0 := +\infty$ and $\mu_{k+1} = -\infty$.) In particular $X_i = F^{\geq\mu_i}X$. The tower has *length* k and we speak of a k*-steps filtration*. Note that this will be later extended (in Subsubsection B.2.1) to allow for non strict inclusions.

Since we consider only *finite* **Q**-filtrations, everytime we deal with a finite number of filtered objects[7], only finitely many breaks occur in the whole, so we could as well tackle them in the framework of **Z**-filtrations. Therefore, we shall feel free to use the results of [5, I.1], which involve **Z**-filtrations.

We then denote $F^{>\mu}X$ the biggest $F^{\geq\nu}X$ for $\nu > \mu$, and $F^{(\mu)}X := F^{\geq\mu}X/F^{>\mu}X$ the successive quotients, so that (with the tower notation):

$$F^{(\mu)}X = \begin{cases} X_i/X_{i-1} \text{ if } \mu = \mu_i, \\ 0 \text{ if } \mu \notin \{\mu_1, \ldots, \mu_k\}. \end{cases}$$

Then to the **Q**-filtered object X is associated a **Q**-graded object:

$$\mathrm{Gr}X := \bigoplus_{i=1}^{k} X_i/X_{i-1} = \bigoplus_{\mu\in\mathbf{Q}} F^{(\mu)}X.$$

B.1.2 Morphisms of filtered objects

A *morphism* of filtered objects X, Y is a morphism $f : X \to Y$ such that $f\left(F^{\geq\mu}X\right) \subset F^{\geq\mu}Y$ for all μ. It induces naturally a morphism of **Q**-graded objects $\mathrm{Gr}f : \mathrm{Gr}X \to \mathrm{Gr}Y$, so we have a covariant functor from **Q**-filtered objects to **Q**-graded objects. We shall respectively denote $\mathrm{Fil}_{\mathbf{Q}}(\mathscr{A})$, resp. $\mathrm{Grad}_{\mathbf{Q}}(\mathscr{A})$, the categories of **Q**-filtered, resp. **Q**-graded objects of $\mathscr{A}$ and Gr the graduation functor.

The morphism $f : X \to Y$ is *strict* (more properly, *strictly compatible with the filtrations*) if:

$$\forall\mu \in \mathbf{Q}\ ,\ f\left(F^{\geq\mu}X\right) = f(X) \cap F^{\geq\mu}Y.$$

Then, if X' is a subobject of the object X of $\mathscr{A}$ (*i.e.* not taking in account the filtration on X) and $X'' := X/X'$,

- There is a unique filtration on X' such that $X' \to X$ is a strict morphism of filtered objects; this *induced filtration* is given by the formula:

$$F^{\geq\mu}X' := X' \cap F^{\geq\mu}X.$$

- There is a unique filtration on X'' such that $X \to X''$ is a strict morphism of filtered objects; this *quotient filtration* (or *coinduced* filtration) is given by the formula:

$$F^{\geq\mu}X'' := F^{\geq\mu}X \pmod{X'}, \textit{i.e.}\text{ the image of } F^{\geq\mu}X \text{ by } X \to X''.$$

In particular, if $f : X \to Y$ is a morphism of filtered objects, its kernel K and cokernel C (as objects of $\mathscr{A}$) respectively inherit filtrations from the source and target of f; and the resulting filtered objects K and C are a kernel and a cokernel of f in $\mathrm{Fil}_{\mathbf{Q}}(\mathscr{A})$.

[7]Such a filtered object can be denoted (X, F) (as for instance in [5, I.1]), but since another filtered object will be denoted (Y, F), it seems simpler to call them X, Y, and so will we do, except when we want to distinguish the filtered object (X, F) from the "plain" object X of $\mathscr{A}$.

Remark B.1. Filtered objects of $\mathscr{A}$ (with the above restrictions on filtrations) form an additive category with finite limits (direct or inverse), in particular kernels, images, cokernels and coimages; but $\mathrm{Fil}_{\mathbf{Q}}(\mathscr{A})$ is not an abelian category. For instance, if an object X of $\mathscr{A}$ is endowed with two distinct filtrations F, G such that $F^{\geq\mu}X \subset G^{\geq\mu}X$ for all μ, then Id_X is a morphism of filtered objects $(X, F) \to (X, G)$; it is a monomorphism and an epimorphism but not an isomorphism. More generally, for a given morphism f of filtered objects, the canonical morphism $\mathrm{Coïm}\, f \to \mathrm{Im} f$ is not necessarily an isomorphism.
On the other hand, if f is strict, then $\mathrm{Coïm}\, f \to \mathrm{Im} f$ is an isomorphism. However, as the following example shows, the compositum of strict morphisms is not necessarily strict, so this is not a way to fix the problem (there is no category of filtered objects with strict morphisms).

Example B.2. We take for $\mathscr{A}$ the category of abelian groups. We endow $X := \mathbf{Z}^2$ with three distinct two-steps filtration F, G, H. For that it is enough to choose in each case the subobject which is neither 0 nor $\mathbf{Z}^2$.
First we let (u, v) a basis of $\mathbf{Z}^2$ and $a \in \mathbf{Z}$ arbitrary (to begin with). We let $L : \mathbf{Z}^2 \to \mathbf{Z}$ the $\mathbf{Z}$-linear form such that $L(u) = a$, $L(v) = 1$. We define:

$$f : \mathbf{Z}^2 \to \mathbf{Z}^2,\ x \mapsto ax \quad \text{and } g : \mathbf{Z}^2 \to \mathbf{Z}^2,\ x \mapsto L(x)u.$$

As intermediate subobjects of $\mathbf{Z}^2$ respectively corresponding to the three two-steps filtration F, G, H, we take $\mathbf{Z}u$, $\mathbf{Z}u$ and $\mathbf{Z}au$. Then one checks easily that f and g are strict morphisms $(\mathbf{Z}^2, F) \to (\mathbf{Z}^2, G)$ and $(\mathbf{Z}^2, G) \to (\mathbf{Z}^2, H)$ but that $g \circ f : (\mathbf{Z}^2, F) \to (\mathbf{Z}^2, H)$ is not strict if $a \geq 2$. It may be helpful to use the following diagram:

$$\begin{array}{ccccc}
0 & \hookrightarrow & \mathbf{Z}u & \hookrightarrow & \mathbf{Z}^2 \\
 & & \downarrow{\scriptstyle f_{|\mathbf{Z}u}} & & \downarrow{\scriptstyle f : x \mapsto ax} \\
0 & \hookrightarrow & \mathbf{Z}u & \hookrightarrow & \mathbf{Z}^2 \\
 & & \downarrow{\scriptstyle g_{|\mathbf{Z}u}} & & \downarrow{\scriptstyle g : x \mapsto L(x)u} \\
0 & \hookrightarrow & \mathbf{Z}au & \hookrightarrow & \mathbf{Z}^2
\end{array}$$

The following properties come from [5, Proposition 1.1.11].

Proposition B.3. *(i) The morphism $f : X \to Y$ is strict if, and only if, the following sequence is exact:*

$$0 \to GrKerf \to GrX \to GrY \to GrCokerf \to 0.$$

(ii) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ an exact sequence, the morphisms f, g being strict. Then the following sequence is exact:

$$GrX \xrightarrow{Grf} GrY \xrightarrow{Grg} GrZ.$$

(The corresponding statement in loc. cit. *is actually more general.)*

B.1.3 Tensor products of filtered objects

Adapter ici [5, 1.1.12]

B.2 Towers and short exact sequences

We consider more concrete (computational) ways of dealing with filtrations. They will be handy when we look at the behaviour of filtered objects under application of an exact functor in Subsection B.3.

B.2.1 k-steps towers

A *k-steps tower* of an object X of $\mathscr{A}$ is an ascending sequence:

$$0 = X_0 \subset \cdots \subset X_k = X$$

of subobjects. So we do *not* require that the inclusions be strict. One reason to so extend the previous definition is that, when we have a morphism a filtered objects, the breaks may not be the same for source and target. So the rule is: to a filtered object (X, F), we associate a tower as above such that for each break μ, the subobject $\mathscr{F}^{\geq\mu}X$ appears as one of the X_i.

Écrire de façon plus précise comment un morphisme de filtrations se transforme en morphisme de tours.
Whenever we have a morphism $f : (X, F) \to (Y, F)$ in $\mathrm{Fil}_{\mathbf{Q}}(\mathscr{A})$, we associate to it k-steps towers on X and Y (here k may be any integer at least equal to the total number of breaks of X and Y); and here the rule is: for every $i = 1, \ldots, k$, there is a μ such that $X_i = \mathscr{F}^{\geq\mu}X$ and $Y_i = \mathscr{F}^{\geq\mu}Y$. Then the condition that f is a morphism in $\mathrm{Fil}_{\mathbf{Q}}(\mathscr{A})$ translates into: it is a morphism $f : X \to Y$ in $\mathscr{A}$ such that $f(X_i) \subset Y_i$ for all i.

Strictness of f then translates easily to the condition that $f(X_i) = f(X) \cap Y_i$ for all i. Not by the way that this condition is empty for $i = 0$ and for $i = k$.

Also we can associate to a k-tower on X a graded object $\bigoplus_{i=1}^{k} X_i/X_{i-1}$. We define in the obvious way the k-tower induced on a subobject $X' \subset X$, resp. coinduced an a quotient object X/X', etc. All results of Subsection B.1 are easily adapted to k-towers.

Remark B.4. We do not try to make k-towers a category (and the above process a functor) because the choice of k depends on the context and there is no natural definition of morphisms between towers of different lengths.

B.2.2 Two steps towers as short exact sequences

We call *short exact sequences* those of the form $0 \to M' \to M \to M'' \to 0$, in particular when $M' \subset M$ and $M'' = M/M'$. A two-steps filtration has the form $0 \subset X_1 \subset X$ and, writing $X' := X_1$, $X'' := X/X_1$, it yields a short exact sequence $0 \to X' \to X \to X'' \to 0$. Let us denote such short exact sequences as $\underline{M}, \underline{X}$.

The language of short exact sequences is therefore to some extent an adaptation of the language of filtered objects. For instance the associated graded object is $\mathrm{Gr}\underline{M} := M' \oplus M''$ and a morphism $\underline{f} := (f', f, f'') : \underline{M} \to \underline{N}$ of short exact sequences is a commutative diagram:

(4)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

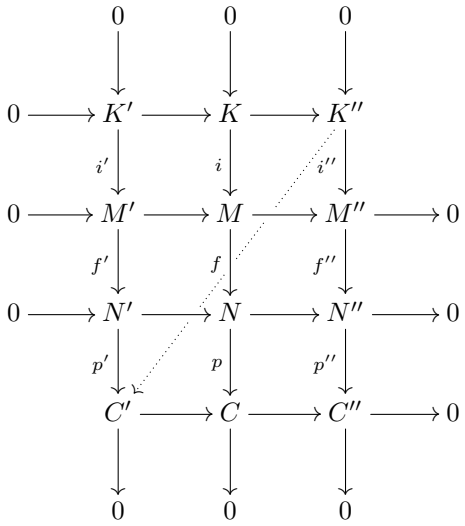
Definition B.5. The morphism $\underline{f} := (f', f, f'') : \underline{M} \to \underline{N}$ is *strict* if:

$$f(M') = f(M) \cap N'.$$

This exactly translates the strictness property of the corresponding morphism of two-steps filtrations. The conditions at $M_0 = 0$ and $M_2 = M$ are empty, one just uses the condition at $M_1 = M'$.

Let $\underline{f} := (f', f, f'') : \underline{M} \to \underline{N}$ a morphism of short exact sequences. Denoting K', K, K'', resp. C', C, C'' the kernels, resp. the cokernels of f', f, f'', we can complete the diagram (4) into a bigger diagram where all displayed sequences (horizontal or vertical) are exact:

(5)



Moreover, by the *snake lemma* [3, §1 no 2,Proposition 2], [14, VIII.4, Lemma 5], [8, II §5 Exercise 7], there is a well defined *connecting homomorphism* $\delta : K'' \to C'$ such that the following sequence is exact:

$$0 \to K' \to K \to K'' \xrightarrow{\delta} C' \to C \to C'' \to 0.$$

Lemma B.6. *The following conditions are equivalent:*
(i) The morphism (f', f, f'') is strict.
(ii) The sequence $0 \to K' \to K \to K'' \to 0$ is exact.
(iii) The sequence $0 \to C' \to C \to C'' \to 0$ is exact.
(iv) The connecting homomorphism δ is null.

Proof. In view of the exact sequence $K \to K'' \xrightarrow{\delta} C' \to C$, the equivalence of the last three conditions is obvious.
Now a bit of diagram chasing shows that, up to natural identifications:

$$\mathrm{Ker}(C' \to C) = \frac{f(M) \cap N'}{f(M')},$$

whence the equivalence of (iii) with (i). □

Remark B.7. Let $\tilde{K}'' := \mathrm{Coker}(K' \to K)$ and $\tilde{C}' := \mathrm{Ker}(C \to C'')$, so that we have exact sequences $0 \to K' \to K \to \tilde{K}'' \to 0$ and $0 \to \tilde{C}' \to C \to C'' \to 0$.

Then there is a monomorphism $\tilde{K}'' \to K''$ and strictness is equivalent to it being an isomorphism. Similarly there is an epimorphism $C' \to \tilde{C}'$ and strictness is equivalent to it being an isomorphism.

B.2.3 Towers as sequences of short exact sequences

To a tower $\underline{M} : 0 = M_0 \subset \cdots \subset M_k = M$ in $\mathscr{A}$, we can associate k short exact sequences $0 \to M_{i-1} \to M_i \to P_i \to 0$, where we denoted $P_i := M_i/M_{i-1}$, $i = 1, \ldots, k$, the successive quotients (so that $\mathrm{Gr}\underline{M} = P_1 \oplus \cdots \oplus P_k$). Let us call (provisionally) $SES_i(\underline{M})$, $i = 1, \ldots, k$, those short exact sequences. Conversely the tower $\underline{M}$ can be recovered from the $SES_i(\underline{M})$.

Let $f : \underline{M} \to \underline{N}$ a morphism of k-towers. It induces morphisms of short exact sequences $f_i : SES_i(\underline{M}) \to SES_i(\underline{N})$. Moreover, f is strict if, and only if, all f_i are strict (this is easy to check and left to the reader).

C'est imprécis. Il faut formuler une équivalence de catégories (additives ...) entre k-tours et k-uples de suites exactes courtes.

B.3 Exact functors and filtered objects

Let $\mathscr{A}, \mathscr{B}$ abelian categories and F an exact functor from $\mathscr{A}$ to $\mathscr{B}$.

B.3.1 Exact functors and short exact sequences and towers

Let $0 \to M' \to M \to M'' \to 0$ a short exact sequence in $\mathscr{A}$. Then $0 \to F(M') \to F(M) \to F(M'') \to 0$, is a short exact sequence in $\mathscr{B}$. A morphism of short exact sequences f in $\mathscr{A}$ is then transformed into a morphism of short exact sequences $F(f)$ in $\mathscr{B}$.

Lemma B.8. *(i) If f is strict, then $F(f)$ is strict.*
(ii) If F is moreover faithful, the converse is true.

Proof. This follows immediately from Lemma B.6. □

A k-tower $\underline{M} : 0 = M_0 \subset \cdots \subset M_k = M$ in $\mathscr{A}$ gives rise to a k-tower $F(\underline{M}) : 0 = F(M_0) \subset \cdots \subset F(M_k) = F(M)$ in $\mathscr{B}$. If the tower $\underline{M}$ decomposes in short exact sequences $SES_i(\underline{M})$ as in Subsection B.2.3, then $F(\underline{M})$ decomposes in short exact sequences $SES_i(F(\underline{M})) = F(SES_i(\underline{M}))$.

Corollary B.9. *(i) If $f : \underline{M} \to \underline{N}$ is a strict morphism, then so is $F(f) : F(\underline{M}) \to F(\underline{N})$.*
(ii) If F is moreover faithful, the converse is true.

Proof. Combine Subsection B.2.3 with Lemma B.8. □

B.3.2 Exact functors and filtered objects

The functor F from $\mathscr{A}$ to $\mathscr{B}$ gives rise to a functor from $\mathrm{Fil}_{\mathbf{Q}}(\mathscr{A})$ to $\mathrm{Fil}_{\mathbf{Q}}(\mathscr{B})$ that we again denote F. From the results of Subsection B.3.1, we deduce the following.

Theorem B.1. *(i) If f is a strict morphism in $Fil_{\mathbf{Q}}(\mathscr{A})$, then so is $F(f)$ in $Fil_{\mathbf{Q}}(\mathscr{B})$.*
(ii) If F is moreover faithful, the converse is true.

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