Appendices

Contents

A The proalgebraic hull of Z

For general definitions and results on tannakian categories and Tannaka's duality, we used [4] and [3], mainly the former. A much more detailed version of what follows can be found in [7].

A.1 Proof of the assertions in 2.1.3

We consider the category $\mathcal{R} = Rep_{\mathbf{C}}(\mathbf{Z})$ of finite dimensional complex representations of $\mathbf{Z}$. Its objects can be seen as pairs (X, ξ) of a finite dimensional complex vector space X and a linear automorphism ξ of X (the image of 1 by the representation). $\mathcal{R}$ is a $\mathbf{C}$-linear abelian category, and it is endowed with an obvious tensor structure, making it a tannakian category. Duals and internal Homs also are the natural ones.

We note that, as an abelian category, $\mathcal{R}$ is exactly the same as the category $Mod^f(R)$ of finite length modules over the principal ideal domain $R = \mathbf{C}[T, T^{-1}]$. This, of course, does not apply to the tensor structure.

The forgetful functor ω is a fibre functor over $\mathbf{C}$, thus making $\mathcal{R}$ a neutral tannakian category. It is therefore equivalent to the category of representations of the proalgebraic group $Gal(\mathcal{R}) = \underline{Aut}^{\otimes}(\omega)$. This is the proalgebraic hull of $\mathbf{Z}$ and we call it $\mathbf{Z}^{alg}$. In this appendix, we shall compute $\mathbf{Z}^{alg}$. More precisely, we shall find the group $Aut^{\otimes}(\omega)$ of its complex points and then indicate briefly its scheme structure.

A.1.1 Natural transformations of ω

We consider a functorial morphism: $\omega \to \omega$, that is an element of $End(\omega)$. For the moment, we do not assume it to be $\otimes$-compatible. It is defined by giving for each representation (X, ξ) a linear automorphism $\overline{\xi}$ of the underlying vector space X in such a way that, for any representation morphism $u : (X, \xi) \to (Y, \eta)$, that is, for any linear map $u : X \to Y$ such that $u\xi = \eta u$, one has the naturality condition: $u\overline{\xi} = \overline{\eta} u$. Now, applying the naturality condition to the automorphism ξ of (X, ξ), one finds that $\overline{\xi}$ commutes with ξ: in the R-module interpretation, this means that $\overline{\xi}$ is a morphism of R-modules.

Applying the naturality condition to the projections of a product to its components, one sees that $\xi \mapsto \overline{\xi}$ is compatible with the decomposition into blocks (direct sums). Using Jordan decomposition, each (X, ξ) can be decomposed into blocks having a single eigenvalue. Moreover, there is no non trivial morphism between such blocks admitting distinct eigenvalues. We can therefore restrict the study to the subcategory of objects (X, ξ) with the single eigenvalue $\lambda \in \mathbf{C}^*$. These correspond exactly to the R-modules with π_λ-torsion, where π_λ is the prime ideal generated by $T-\lambda$. Jordan blocks correspond to cyclic modules (torsion modules with one generator). Therefore, we can consider only morphisms between cyclic modules.

Each eigenvector x of ξ may be considered as a morphism: $(\mathbf{C}, \lambda) \to (X, \xi)$. Applying the naturality condition shows that $\overline{\xi}(x) = \overline{\lambda} x$. This entails $Sp(\overline{\xi}) = \{\overline{\lambda}\}$.

We then consider (X, ξ) and (Y, η), cyclic modules of ranks n and p, therefore, respectively isomorphic to $\frac{R}{\pi_\lambda^n}$ and $\frac{R}{\pi_\lambda^p}$. Thus, any morphism u between them is a homothety $a \mapsto ay$ and the naturality condition is equivalent to $\overline{\eta}(y) = \overline{\xi}(1)y$ (the multiplication makes sense because of the torsion condition on y). From this, we draw first that the morphism $\overline{\xi}$ attached to the representation (X, ξ) corresponding to the R-module $\frac{R}{\pi_\lambda^n}$ can be identified with an *element* $\overline{\xi_{\lambda,n}} \in \frac{R}{\pi_\lambda^n}$; second, that the image of $\overline{\xi_{\lambda,n}}$ in $\frac{R}{\pi_\lambda^{n-1}}$ is $\overline{\xi_{\lambda,n-1}}$. It is not hard, making explicit all morphisms between cyclic modules, to see that there are no more conditions to be checked.

We conclude that the natural transformation $(X, \xi) \mapsto \overline{\xi}$ of ω is completely described by a compatible family $(\overline{\xi_\lambda})_{\lambda \in \mathbf{C}^*}$ where each $\overline{\xi_\lambda} \in \lim\limits_{\leftarrow} \frac{R}{\pi_\lambda^n}$. This means that $End(\omega)$ is canonically isomorphic to $\hat{R} = \prod\limits_{\lambda \in \mathbf{C}^*} \mathbf{C}[[T-\lambda]]$. Concretely, a representation (X, ξ) gives rise to a finite length R-module, hence to a $\hat{R}$-module (since $\hat{R}$ is the projective limit of all quotient rings $\frac{R}{I}$). Then $\overline{\xi}$ is described by a homothety on each of its torsion components, by an element of the corresponding $\mathbf{C}[[T-\lambda]]$.

A.1.2 Transformations preserving the tensor structure

We now consider a $\otimes$-compatible transformation $(X, \xi) \mapsto \overline{\xi}$ of ω. It is automatically an isomorphism, thus, an element of $Aut^{\otimes}(\omega)$. Clearly, it is enough to exploit this multiplicative condition for Jordan blocks.

The multiplicative condition applied in dimension 1 shows that $\lambda \mapsto \overline{\lambda}$ is a group homomorphism. Then, tensoring $(\mathbf{C}, \lambda)$ with the unipotent Jordan block of rank n gives the the Jordan block of rank n of eigenvalue λ. It therefore suffices to exploit the multiplicative condition on unipotent Jordan blocks.

So call $(\mathbf{C}^n, \xi_{1,n})$ the standard model for the unipotent Jordan block of rank n. "A little plethysm" gives:

$$(\mathbf{C}^n, \xi_{1,n}) \otimes (\mathbf{C}^p, \xi_{1,p}) \simeq \bigoplus (\mathbf{C}^q, \xi_{1,q})$$

where q runs through $\{n+p-1, n+p-3, ..., |n-p|+3, |n-p|+1\}$. From this, one concludes that the element P of $\mathbf{C}[[T-1]]$ that corresponds to our transformation satisfies: $P(U+V+UV) = P(U)P(V)$. It is an easy exercice to show that $P(U) = (1+U)^a$ for some complex number a, so that, for unipotent representations (X, ξ), one has $\overline{\xi} = \xi^a$

(which makes sense for a unipotent automorphism). This achieves the proof of the description of $\mathbf{Z}^{alg}$ given in section 2.1.3. We propose to call the resulting group the "super-proalgebraic hull".

A.1.3 Group scheme structure

The group scheme $Gal(\mathcal{R}) = \underline{Aut}^{\otimes}(\omega)$ represents the functor in $\mathbf{C}$-algebras: $S \rightsquigarrow Aut^{\otimes}(\omega \circ \Phi_S)$ where Φ_S is the functor $M \rightsquigarrow M \otimes_{\mathbf{C}} S$. Call G this group scheme. An element of $G(S)$ is the datum, for each representation (X, ξ), of a S-automorphism $\overline{\xi}$ of $X \otimes S$ with naturality and multiplicative conditions. To make it explicit, we must again consider Jordan blocks, and separate the one dimensional effect from the effect on unipotent blocks. For unipotent objects, one finds that $\overline{\xi} = \xi^a$ for some $a \in S$. This gives the unipotent part of G: it is defined by $G_u(S) = S$, hence it is the affine group G_a.

In dimension 1, one finds a group homomorphism $\lambda \mapsto \overline{\lambda}$ from $\mathbf{C}^*$ to S^*. This gives the semi-simple part: $G_s(S) = Hom_{gr}(\mathbf{C}^*, S^*)$. It is the projective limit of the functors in groups $S \rightsquigarrow Hom_{gr}(H, S^*)$, where H runs through the inductive system of finitely generated subgroups of $\mathbf{C}^*$. For such a subgroup, it is easy to see that the functor is represented by the affine scheme with algebra $\mathbf{C}[H]$, the group algebra of H over $\mathbf{C}$, which has a natural Hopf algebra structure. We conclude that $G_s = Spec\,(\mathbf{C}[\mathbf{C}^*])$.

A.2 Additional informations

A.2.1 Chevalley duality

In [2], Chevalley constructs the complex algebraic group attached to a compact real Lie group from the category of its representations. This can be considered as the true birth of Tannaka duality. The construction is simplified a little by the fact that the representations of a compact group are completely reducible, so that the naturality conditions we used above boil down to compatibility with direct sums and conjugations.

Therefore, one may wonder what that method would give for $\mathbf{Z}$: a bigger group that $\mathbf{Z}^{alg}$, for sure, but how much bigger ? It is proved in [7] (along with an explicit computation) that the group thus obtained has $\mathbf{Z}^{alg}$ as a subgroup of order 2. This is due to the "plethysm" formula above and its consequence that the knowledge of $\overline{\xi_{1,2}}$ only completely determines an element ; and the unique eigenvalue of $\overline{\xi_{1,2}}$ has to be ± 1 (again because of plethysm).

A.2.2 The proalgebraic hull of Z over other base fields

Our computation of $\mathbf{Z}^{alg}$ is easily seen to work the same if one replaces $\mathbf{C}$ by any algebraically closed field of null characteristic.

We now consider a field K of such that $char K = p > 0$. One can use again Jordan reduction and obtain the semi-simple component in a similar way: it is $Hom_{gr}(K^*, K^*)$, and the group scheme structure can be determined as above.

As for the unipotent component, it is determined by the "formal group" we called P, that is, a formal power series such that $P(U+V+UV) = P(U)P(V)$ and $P(0) = 1$. One can show that $P(U) = (1+U)^a$ where a is a p-adic number: $a = \sum k_i p^i \in \hat{\mathbf{Z}}_p$. Then the product $(1+U)^{\sum_{i\geq 0} k_i p^i} = \prod_{i\geq 0}(1+U)^{k_i p^i}$ is U-adically convergent. Thus, our unipotent component is now the profinite discrete group $\hat{\mathbf{Z}}_p$.

It is easy to see that the unipotent components of the proalgebraic hulls of $\mathbf{Z}$ over algebraically closed fields of all characteristics actually are the fibres over the corresponding geometric points of a single scheme over $\mathbf{Z}$. One can naturally ask what the latter classifies. Such a modular interpretation has been given by Bertrand Toen in [10].

A.2.3 Bibliographic and folkloric informations

While the above basic "exercice" does not appear in the litterature, one can find some related information in the following references:

1. The book in preparation of Jean-Pierre Ramis [6] on the inverse problem in differential Galois theory contains a computation of $\mathbf{Z}^{alg}$ by a more conceptual method (but with more prerequisites about algebraic groups).

2. In section 5.2. of [5] one finds the detailed description of the algebraic quotients of $\mathbf{Z}^{alg}$ but neither proof nor description of the action.

3. According to Y. André:
En fait Z^{alg} fait partie du folklore, Serre l'avait decrit dans son cours au College sur les motifs, et il attribuait cette description a Chevalley.

4. The paper "Differential equations in characteristic p" (M. van der Put,Compositio Math. 97, 227-251, 1995) section 2.3, contains a result similar in spirit but nevertheless different what is computed there is a kind of additive version of $\mathbf{Z}^{alg}$, with "Lie-like" elements instead of "group-like" elements.

Besides, the latter authors have proposed swifter ways to get the description of $\mathbf{Z}^{alg}$:

1. According to Y. André:
Le point essentiel est le suivant: soit $u \in GL_n(K)$, K de caracterisitique nulle. Soit A_u le plus petit sous-groupe algebrique de GL_n contenant u. Alors si u est semi-simple, A_u est un groupe de type multiplicatif, de torsion cyclique (son groupe des caracteres est le sous-groupe de $\bar{K}^*$ engendre par les valeurs propres de u). Si u n'est pas semi-simple, on considere sa "decomposition de Dunford" dans GL_n, et la partie unipotente donne lieu a un facteur additif G_a dans A_u. Z^{alg} est fabrique par limite projective filtrante, et se decompose en $(G_a) \times (\mu_\infty) \times$ un protore.

2. According to M. van der Put:
For an algebraically closed field of characteristic 0, one considers $H := k^*$ as abstract group and forms its group ring $\mathbf{Z}[H]$. Then $G := Spec(\mathbf{Z}[H]) \times \mathbf{G}_a = Spec(\mathbf{Z}[H][t])$ is an affine group scheme over $\mathbf{Z}$. One has to show that there are "canonical" isomorphisms $G(R) \to \mathrm{Aut}^{\otimes}(\omega)(R)$ for any k-algebra R (i.e., an isomorphism of the functors G and $\mathrm{Aut}^{\otimes}(\omega)$ on the category of all k-algebras). In the

case that k is an algebraically closed field of characteristic $p > 0$, the group scheme G is a little different, namely $G = Spec(\mathbf{Z}[H]) \times \hat{\mathbf{Z}}_p$, where H denotes k^* as abstract group and $\hat{\mathbf{Z}}_p$ denotes the constant group scheme over $\mathbf{Z}$, corresponding to the abstract group $\hat{\mathbf{Z}}_p$. Again one has to show that the two functors G and $\mathrm{Aut}^{\otimes}(\omega)$ are isomorphic as functors on the category of all k-algebras.

B Automorphisms of the "field of solutions"

We compute here a group that bears similarities at the same time with our Galois group and with the group computed by van der Put and Singer in [5], chap. 12, pp. 150-151 (we detail in [7] the important differences of these two groups). This is the closest we get from Picard-Vessiot theory ! Moreover, some aspects of the following computation may shed some light on the factor g_a that helped building the "mistigri" ψ_a in 3.2.

B.1 Fields of solutions

We improperly call, in the title of this appendix, "field of solutions" the extension:

$$\mathbf{K} = \mathbf{k}(l_q, (e_{q,c})_{c \in \mathbf{C}^*})$$

of $\mathbf{k} = \mathbf{C}(z)$ obtained by adding the basic construction tools needed to build solutions. Note however that we rather seeked the said solutions in the field $\mathbf{K}_0 = \mathbf{K} \otimes_{\mathbf{k}} \mathbf{k}_0$ where $\mathbf{k}_0 = \mathcal{M}(\mathbf{C})$. Since the characters are enough to generate all elliptic functions, it is plain that actually:

$$\mathbf{K} = \mathcal{M}(\mathbf{E})(l_q, (e_{q,c})_{c \in \mathbf{C}^*}).$$

It is a subfield of $\mathcal{M}(\mathbf{C}^*)$, invariant under the automorphism σ_q. We shall compute the group:

$$G = \{\phi \in Aut(\mathbf{K}/\mathbf{k}) \ / \ \phi \circ \sigma_q = \sigma_q \circ \phi\}.$$

Here are three basic facts:

1. An element $\phi \in G$ leaves invariant $\mathbf{K}^{\sigma_q} = \mathcal{M}(\mathbf{E})$.
2. It sends a non zero c-character f to a non zero c-character, so that then, $\frac{\phi(f)}{f} \in \mathbf{K}^{\sigma_q}$.
3. Similarly, it sends a q-logarithm g to a q-logarithm, so that then $\phi(g) - g \in \mathbf{K}^{\sigma_q}$.

B.2 Automorphisms

B.2.1 G as a fibered product

From basic fact 1 above, one draws that there is a restriction morphism $G \to G_0 = Aut(\mathcal{M}(\mathbf{E})/\mathbf{C})$; from basic facts 2 and 3, that the fields $\mathcal{M}(\mathbf{E})((e_{q,c})_{c \in \mathbf{C}^*})$ and $\mathcal{M}(\mathbf{E})(l_q)$ are invariant under G. Define:

$$\begin{cases} G_1 = \{\phi \in Aut(\mathbf{K_1}/\mathbf{k_1}) \ / \ \phi \circ \sigma_q = \sigma_q \circ \phi\} \\ G_2 = \{\phi \in Aut(\mathbf{K_2}/\mathbf{k_2}) \ / \ \phi \circ \sigma_q = \sigma_q \circ \phi\} \end{cases} \quad \text{where} \quad \begin{cases} \mathbf{K_1} = \mathcal{M}(\mathbf{E})((e_{q,c})_{c \in \mathbf{C}^*}) \\ \mathbf{K_2} = \mathcal{M}(\mathbf{E})(l_q) \end{cases}$$

and where $\mathbf{k_i} = \mathbf{K_i} \cap \mathbf{k}$ $(i = 1, 2)$. This gives rise to two morphisms: $G \to G_1$ and $G \to G_2$. From the results in [8], appendix A, one draws:

$$G = G_1 \times_{G_0} G_2.$$

We have to study the two projection morphisms.

From now on, we shall write $\overline{\phi}$ the restriction of an automorphism $\phi \in G$ to $\mathcal{M}(\mathbf{E})$.

B.2.2 Action on the q-logarithm

Writing, for $\phi \in G_2$, $a_\phi = \phi(l_q) - l_q$, one notes that ϕ is determined by the pair $(\overline{\phi}, a_\phi) \in G_0 \times \mathcal{M}(\mathbf{E})$. Since $\psi(\phi(l_q)) = l_q + a_\psi + \overline{\psi}(a_\phi)$, one draws that G_2 is isomorphic to the semi-direct product of G_0 by $\mathcal{M}(\mathbf{E})$. It is not a direct product. We remark that the component $\mathcal{M}(\mathbf{E})$ was invisible in our Galois group.

B.2.3 Action on q-characters

In a similar way, $\phi \in G_1$ is determined by $\overline{\phi}$ and the family of the $f_c = \frac{\phi(e_{q,c})}{e_{q,c}} \in \mathcal{M}(\mathbf{E})^*$. One must have $f_{qc} = f_c$ since $\frac{e_{q,qc}}{e_{q,c}} \in \mathbf{k}$ is fixed by ϕ. We shall compute the image and the kernel of $G_1 \to G_0$.

B.2.3.1 The kernel of $G_1 \to G_0$.- An element of the kernel has $f_c f_d = f_{cd}$, thereby defining a group morphism $\mathbf{C}^* \to \mathcal{M}(\mathbf{E})^*$; composing, we get $\mathbf{C}^* \to \mathcal{M}(\mathbf{E})^*/\mathbf{C}^*$, a morphism from a divisible group to a free one, hence trivial. Since $f_{qc} = f_c$, we have obtained a morphism $\mathbf{C}^* \to \mathbf{C}^*$ such that $q \mapsto 1$. This obviously works the other way round, and we conclude:

$$Ker(G_1 \to G_0) = Hom_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*).$$

B.2.3.2 The structure of G_0.- From the classical theory of elliptic curves (*see* [9]), one draws:

$$G_0 = Aut(\mathbf{E}).$$

This group consists in *changes of variables*: $z \mapsto az$, $z \mapsto \frac{1}{z}$ and maybe , for some special values of q, $z \mapsto z^\imath$ and $z \mapsto z^\rho$.

B.2.3.3 The image of $G_1 \to G_0$.- Studying divisors, one can prove that none of the last three types of change of variables can come from an element of G_1. On the other hand, $z \mapsto az$ is in the image and we now determine its antecedents. We specify such an antecedent by writing $\phi(e_{q,c}) = \frac{e_{q,c}(az)}{g(c)(z)}$. Then we find that g is a morphism $\mathbf{C}^* \to \mathbf{C}^*$ such that $q \mapsto a$. We conclude:

The image G_0' of $G_1 \to G_0$ is made up of the translations $\overline{\phi_a}$, hence isomorphic to $\mathbf{E}$. *The antecedents of $\overline{\phi_a}$ correspond bijectively to the morphisms $\mathbf{C}^* \to \mathbf{C}^*$ such that $q \mapsto a$.*

B.2.3.4 The structure of G_1. - To summarize, any $\phi \in G_1$ can be written:

$$\phi_g : e_{q,c} \mapsto \frac{e_{q,c}(az)}{g(c)}$$

where g is a morphism $\mathbf{C}^* \to \mathbf{C}^*$ such that $q \mapsto a$. Moreover, one finds that: $\phi_g = \phi_h \Leftrightarrow \exists r \in \mathbf{Z} \ : \ \forall c \in \mathbf{C}^* \ , \ g(c) = c^r h(c)$. The group G_1 can therefore be identified to the quotient of $Hom_{gr}(\mathbf{C}^*, \mathbf{C}^*)$ by the subgroup of the $z \mapsto z^r$ $(r \in \mathbf{Z})$. Identifying G'_0 to $\mathbf{E}$, the restriction morphism $G_1 \to G'_0 \subset G_0$ is given by $g \mapsto g(q) \pmod{q^{\mathbf{Z}}}$.

The kernel can be identified to $Hom_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*/q^{\mathbf{Z}})$ quotiented by the subgroup of the $z \mapsto z^r$ $(r \in \mathbf{Z})$. This is isomorphic to $Hom_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*) = \check{\mathbf{E}}$.

C Tamely irregular equations

We study here irregular systems that admit convergent fundamental solutions. We shall moreover assume that no ramification is needed. We intend to return later to the ramified case, which involves some algebraic complications. Here, we shall expound the bare minimum needed to handle some useful applications.

C.1 Convergent solutions

Say that a system has *level* $\nu \in \mathbf{N}^*$ if all the slopes of its Newton polygons at 0 and ∞ belong to $\frac{1}{\nu}\mathbf{Z}$.

C.1.1 The categories $\mathcal{E}_1$ and $\mathcal{E}_{t,1}$

From the general properties of Newton polygons alluded in 1.2.1, it follows that the full subcategory of $\mathcal{E}$ whose objects have level 1 is a tannakian subcategory ; we call it $\mathcal{E}_1$.

We know each object of $\mathcal{E}_1$ to be formally equivalent at 0 to a direct sum of pure objects, and the same at ∞. We call such an object *tamely irregular* (of level 1) if these equivalences can be realized through meromorphic gauge transformations. This again defines a tannakian subcategory $\mathcal{E}_{t,1}$ of $\mathcal{E}_1$. We shall see it to be neutral.

C.1.2 Local fundamental solutions and connection matrix

An equation of $\mathcal{E}_{t,1}$ has local solutions $X^{(0)}$ and $X^{(\infty)}$ that are direct sums of "pure" solutions, of the form $M\Theta^\mu e_{q,A}$, where μ runs through the slopes, M is meromorphic and A is constant invertible. Then the connection matrix $P = \left(X^{(\infty)}\right]^{-1} X^{(0)}$ is elliptic.

The same construction for a level ν equation would also have given rise to an elliptic matrix, but with respect to another elliptic curve, isogenous to $\mathbf{E}$.

C.2 Tame connection data

To the tamely irregular equation A of level 1, we associate a *connection triple*:

$$\left(\sum_{\mu \in \mathbf{Z}} z^\mu A_\mu^{(0)}, P, \sum_{\mu \in \mathbf{Z}} z^\mu A_\mu^{(\infty)} \right).$$

C.2.1 The category $\mathcal{C}_{t,1}$

As before, we thus form an abelian tensor category, that we call $\mathcal{C}_{t,1}$. We do that just by adding the rule $z^\mu A \otimes z^{\mu'} A' = z^{\mu+\mu'} A \otimes A'$. The same arguments as before show $\mathcal{C}_{t,1}$ to be equivalent to $\mathcal{E}_{t,1}$: indeed, we may again use properties of fundamental solutions with coefficients in the field $\mathcal{M}(\mathbf{C}^*)$.

C.2.2 The groups $G_{t,1}^{(0)}$ and $G_{t,1}^{(\infty)}$

With respect to the fuchsian case, the main difference is that the local components are now $\mathbf{Z}$-graded. This means that the local Galois groups inherit a factor $\mathbf{C}^*$:

$$\begin{cases} G_{t,1}^{(0)} = G^{(0)} \times \mathbf{C}^* \\ G_{t,1}^{(\infty)} = G^{(\infty)} \times \mathbf{C}^* \end{cases}$$

For a specific equation, an element λ of the factor $\mathbf{C}^*$ acts by λ^μ on a block of slope μ ; in particular, its action is trivial for the slope 0.

C.2.3 Devissage and Stokes operators

We emphasize the following interesting consequence of Adam's lemma. Since the first slope (say, at 0) admits a full complement of convergent solutions, any irregular system admits a devissage by tamely irregular equations.

Now, suppose one has found a local fibre functor that restricts to $\omega^{(0)}$ on tamely irregular equations. Then there is a restriction morphism of the local Galois groups: $G_{i,1}^{(0)} \to G_{t,1}^{(0)}$. It is natural to consider as the Stokes operators caused by the irregularity at 0 the elements killed by this morphism. But it follows from the above devissage that the kernel of this morphism is unipotent.

More generally, in the local category, every irregular object comes with a canonical filtration by the slopes such that the quotients are pure. We can therefore attach to it its "graded module", a tamely irregular object. The fact that a slope is a quotient of a degree by a rank suggests a strong analogy with the filtration of a vector bundle by semi-stable bundles, although, strangely, the slopes are here in the reverse order.

C.3 Examples

We just want to show that tamely irregular equations spontaneously appear in nature, thereby motivating a special study. We shall not treat the examples in great detail.

C.3.1 An example arising from the bispectral problem

This example was communicated to us by Emil Horozov. It comes from the theory of bispectral operators, *see* [1], appendix A. It arises when one looks for fixpoints of the so-called "q-deformed generalized Bessel operators". In our notations, the equation is:

$$\left[\left(\sigma_q - q^{\beta_1}\right) \cdots \left(\sigma_q - q^{\beta_N}\right) - z^N \right] f = 0.$$

Note that it contains as a particular case the equation satisfied by the q-Bessel function $J_\nu^{(1)}(x;q)$ studied by Changgui Zhang in [11] (see equation 0.3 there).

The equation is clearly fuchsian at 0, with exponents the q^{β_i}, so that its local Galois group at 0, depends only on the multiplicative relations of these exponents modulo q.

At ∞, it is pure of slope 1, hence tamely irregular. Writing

$$(T - q^{\beta_1}) \cdots (T - q^{\beta_N}) = \sum_{i=0}^{N} \alpha_i T^{N-i}$$

the characteristic equation, putting $w = \frac{1}{z}$, $f(z) = g(w) = \Theta_q(w)h(w)$, one has the equation:

$$h(q^N w) + \sum_{k=0}^{N-1} \frac{(-1)^{k-N} q^{\frac{k(k+1)-N(N+1)}{2}} \alpha_k w^k}{\alpha_N w^N - q^{-N^2}} h(q^k w) = 0.$$

This is fuchsian, with exponents the $(-q^{\frac{N+1}{2}} e^{\frac{2\imath\pi k}{N}}), 0 \leq k \leq N-1$. Thus, the local Galois group at ∞ is made up of a $\mathbf{C}^*$ factor another factor, namely the fuchsian local Galois group of the latter equation.

C.3.2 An example from q-integrability theory

This example was communicated to us by Alfred Ramani. It comes from the work of Grammatikos and Ramani on q-integrability, in theoretical physics.

One linearises the homographic equation $\sigma_q f = \frac{f-az}{c(f-c)}, ac \neq 0$ by putting $f = \frac{P}{Q}$, where:

$$\sigma_q \begin{pmatrix} P \\ Q \end{pmatrix} = A \begin{pmatrix} P \\ Q \end{pmatrix}, \quad \text{where:} \quad A(z) = h(z) \begin{pmatrix} 1 & -az \\ c & -c^2 \end{pmatrix}.$$

The choice of h is arbitrary. It can change the fuchsianity of the system at 0 or ∞, but it cannot change the tameness: indeed, the presence of h amounts to multiplying solutions by a function H such that $\sigma_q H = hH$, and this does not produce divergent series.

We take $h = 1$, whence a system fuchsian at 0. The equation at ∞ is given by the matrix:

$$A\left(\frac{1}{qw}\right)^{-1} = \frac{q}{ac - qc^2 w} \begin{pmatrix} -c^2 w & \frac{a}{q} \\ -cw & w \end{pmatrix}.$$

Again, the functional factor has no effect on tameness and we get rid of it.

Now, for any system of rank 2: $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, an equivalent equation of order 2 is:

$$\sigma_q^2 F - \left(\sigma_q(\alpha) + \delta \frac{\sigma_q(\beta)}{\beta}\right) \sigma_q F + \frac{\sigma_q(\beta)}{\beta} (\alpha\delta - \beta\gamma) F = 0.$$

We therefore find the equation:

$$F(q^2 w) + (qc^2 - 1) w F(qw) + cw \left(-cw + \frac{a}{q}\right) F(w) = 0.$$

This is pure of slope $\frac{1}{2}$, hence tamely irregular of level 2.

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