Basic Modern Theory of Linear Complex Analytic q-difference equations

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ABSTRACT. This book is about q-difference equations. It focuses on techniques inspired by differential equations, in the straight line of Birkhoff's 1913 Master Work, as revitalized in the last three decades. It follows the approach of the Ramis school, mixing algebraic and analytic methods. Although it uses some q-calculus and is illustrated with the help of q-special functions, neither of those are its main object. Nine appendices aim to help the reader with some fundamental but not universally taught facts.

This book is dedicated to Iannis.

Contents

Foreword

There are, if we restrict to functions in a single variable, three big families of functional equations:

- the differential equations connecting a function f to its successive derivatives f', f'', $\ldots$;
- the difference equations connecting a function f to its successive translates $f(x+a)$, $f(x+2a)$, $\ldots$;
- the q-difference equations connecting a function f to its successive "dilatations" $f(qx)$, $f(q^2x)$, $\ldots$.

They are related to homographies of the Riemann sphere and to their infinitesimal counterpart. I shall only speak of linear equations with coefficients that are polynomial or, more generally, holomorphic. The subject of the book of Jacques Sauloy is the third case, that of analytic linear q-difference equations.

Discretizations of differential equations give difference and q-difference equations with constant coefficients. One is then interested in their discrete solutions. But it is more interesting to consider their meromorphic solutions. We may for instance consider the solution $(n-1)!$ of the discrete equation $f(n+1) = nf(n)$, but it is much more interesting to look at the solution $\Gamma(x)$ of $f(x+1) = xf(x)$, for x real, like Euler, or better for x complex. J. Sauloy studies in his book meromorphic solutions[1] of q-difference equations.

The history of the subject begins with q-calculus, based on q-deformations of integers like: $n \mapsto [n]_q = \frac{q^n-1}{q-1}$, and of functions: $f(x) \mapsto f(qx)$. Later appear q-special functions. In an enthralling historical prelude, J. Sauloy shows the birth and development from Fermat and Euler to Watson. The formalized notion of q-difference equation and the separation of q-calculus and equations only appear late: in the work of Jackson in 1910[2]. In the case of differential and difference equations, the notion of operator had been extracted from the gangue of computations by the alsacian mathematician Louis François Antoine Arbogast more than a century before, in 1800.

It is interesting to note that, even recently, most experts in q-calculus, q-orthogonal polynomials, q-special functions or q-combinatorics did not relate those questions to q-difference equations. Ramanujan does not speak of q-difference equations, but neither do Gasper and Rahman in the first edition [3] of their "bible" on q-hypergeometric functions (although the link was clearly established by Jackson

[1]The "great elders", in particular G. D. Birkhoff, considered also ramified solutions, but I proposed around 1990 a change of paradigm by limiting oneself to meromorphic solutions.

[2]After however works on functional equations in France at the end of XIXth century, in particular by Picard and Poincaré, but with a different point of view.

[3]The expression "q-difference equations" appears marginally in two places in the second edition.

in his work on Heine's q-series ...). Richard Askey signalled the importance of the point of view of q-difference equations, but rather tardily. J. Sauloy shows in his book how that point of view allows one to transform the jungle of q-calculus and q-special functions into a harmonious garden *à la française*.

When I was a student in the Sorbonne in Paris, at the beginning of the sixties, our masters thought that the history of analytic differential equations had come to an end in the late twenties[4]. But there was a vigorous revival at the beginning of the seventies, with in particular a return to Birkhoff, whose works on the subject had been completely forgotten. New concepts and techniques appeared (or reappeared): Riemann-Hilbert correspondence, Newton polygons, Gevrey asymptotic expansions, k-summability, index theorems, Stokes multipliers, sheaf cohomology, differential Galois theory, density theorems .... An essential point is the treatment of irregular equations. Later a modern parallel theory of q-difference equations followed[5]. The goal of the book by J. Sauloy is to present the foundations of that theory. The book contains in particular the classification of *regular singular* equations by crossing analytic and Galois theoretic points of view. There also is an introduction to the irregular case, which, as in the differential case, is much more difficult. Thus appear the notions of q-asymptotics, summability and q-Stokes phenomenon[6].

Recently some q-difference equations (possibly non linear, like the q-analogues of Painlevé equations) appeared in a large number of works in theoretical physics[7]. For instance, theoretical physicists think that a "q-deformation" is involved when going from "two dimensional conformal field theory" to the theory in dimension three[8]. Some q-difference equations also appear in many works by Andrei Okounkov, in relation with various subjects: "enumerative geometry, string theory, mirror symmetry, quantum gravitation".

Given the important development of q-difference equations in various domains of mathematics and theoretical physics, the lack of a basic work, crossing various approaches, was a huge lacuna. Everything was scattered in various works or research articles, including the "classical" basics[9]. The book by J. Sauloy perfectly fills this gap. I am convinced that it will become an essential tool for a whole community of researchers verging on the q-world in their work, and a very good entry point for students. it should establish itself over the years as the reference book on the subject.

Jean-Pierre Ramis, member of the French Academy of Sciences

[4]It is clearly the point of view of J. Dieudonné in his "Abrégé d'histoire des mathématiques".

[5]The roots of the modern theories of differential and q-difference equations lie in great part in a 1913 article by Birkhoff who tackles the three "sister theories" of differential, difference and q-difference equations from a point of view which J. Sauloy calls Riemann-Hilbert-Birkhoff theory.

[6]The author announces that those subjects (and others ...) will be deepened in a companion work in preparation.

[7]One even meets in some works the q-Stokes phenomenon.

[8]According to Beem, Dimofte and Pasquetti "the natural 3D analogues of the differential equations whose solutions determine the partition function in two-dimensions are q-difference equations".

[9]There did not even exist for q-differences the analogue of Batchelder's 1930 book for differences.

Preface

In 2003, Jean-Pierre Ramis (of whom I had been a student), Changgui Zhang (who also had been a student of Ramis), Lucia Di Vizio (who had been a student of Yves André) and myself together wrote a short survey article on q-difference equations for "La Gazette des Mathématiciens", a publication of the Société Mathématique de France. The theory of q-difference equations had, in the preceding years, known a revival under (notably) the impulsion of such as Jean-Paul Bézivin, Hidetaka Sakai, Marius van der Put, Michael Singer, Yves André, and mostly Ramis himself.

Following our article, Yves André suggested that a team of valiant involved mathematicians should write a modern book on the subject. So the same group, under the wise direction of Ramis, begun to prepare it. After (I believe) two years of false starts and procrastination, the project was abandonned. The life of "enseignants-chercheurs" in France does not favour long term projects. Moreover, as Hardy said, "young mathematicians should prove theorems, old mathematicians should write books". (It is true that in the meantime I produced a book , but it was a more or less direct adaptation of courses I had given on very classical matters.)

Four years ago, the beginning of my retirement and the Covid pandemic compelled me to resume the project. But diminution of task force (to 25 %) and difference of temper (see below) make the result quite different of what it would have been. An obvious reason is my narrower domain of expertise. The scope of the resulting book is certainly narrower to the same extent. Another one is that, as "enseignant-chercheur", I probably have a bigger proportion in me of "enseignant" and a lesser of "chercheur" than my worthy colleagues. So the book is certainly more didactic and less systematic that it could have been. I favour examples, motivation, analogies, historical reminders, exercises in numbers, redundancy[10] ... Besides, all along, I wrote this book with the "Graduate Studies in Mathematics" in mind, until the AMS assessed that it should rather belong to the "Mathematical Surveys and Monographs" series.

Also, I am more attracted by tools and techniques than by problems. In some sense, in Marx classification, this book belongs to the sector I (where are produced the instruments of production). I have been (unhappily ?) encouraged in this direction by my activity of referee for many interesting articles in the last two decades, which tried to use the nascent modern theory of q-difference equations but revealed misunderstandings of its basics, although the attempt at using it was

[10]Curious readers may notice that in many places my presentation of facts is longer than in the original articles they come from. Let me quote Johannes Huebschmann in the december 2023 issue of AMS Notices: "The diligent reader will notice that the proof is complete *i.e.*, no detail is left to the reader. It is an instance of the common observation to the effect that *mathematics consists in continuously improving notations and terminology.*" (Emphasis in the source.)

plainly justified and the rest of the articles was valuable. The reader will easily spot other biases of mine (no Cauchy integral, function theory almost entirely through power series, etc).

For all the shortcomings of this book I can only propose a compensation: a kind of "companion volume" with broader scope and at a higher level of sophistication will be produced within a reasonable delay (say two years), tackling important omissions (q-asymptotics and summation, q-special functions, q-Painlevé equations, various aspects of Galois and of moduli theory, phenomena related to non integral slopes, effective methods, solutions of q-difference equations *vs* solutions of differential equations ...); this will be a collective work by Shaoshi Chen, Yousuke Ohyama, Jean-Pierre Ramis, Julien Roques, Changgui Zhang and myself.

There is another shortcoming of the book[11] for which I can only hope that future work (presumably by others) will bring reparation. During the decades I devoted to q-difference equation theory, I tumbled upon a mass of facts for which an underlying order could be guessed, and here, I have tried to make that order explicit. There are quite a few places[12] where I think more order can be expected.

Paying dues. It is time to thank those who helped so much: above all, Jean-Pierre Ramis, of course; my colleagues, friends and coauthors Lucia Di Vizio, Charlotte Hardouin, Yousuke Ohyama, Julien Roques and Changgui Zhang; my colleagues, friends and commensals Anne Duval, Michèle Loday, Claudine Mitschi, Vadim Schechtman and Joseph Tapia; and a special word for Yannis with whom it all begun.

I should add the six anonymous colleagues who read the manuscript and made helpful suggestions. And, above all, "my" AMS editor, Ina Mette, who gave me all help and encouragement I needed all along this lengthy affair.

Dedication. I dedicate this book to Yannis Varouchas (1950-2003), also known as Jean Varouchas. Of greek origin, he came to study in France in 1969, when I met him in the "hypotaupe" of André Warusfel. When we started (under the guidance of Georges Maltsiniotis) to meet "true math", he expressed strong dislike of mathematics based on arrows (as he said), but later was reconciled with algebraic geometry by the book of Griffiths and Harris. As a devoted teacher, he despised the formalist tendency and said that after three years in University, french students only knew three functions: exp, log and f. In the last years before his untimely death, he caught the q-disease, which was a happy event for me since it allowed us to meet anew.

Obituary. Bernard Malgrange (1928-2024) passed away on january 5^{th}, just after I had finished writing this book. I reproduce further below the few lines (in french, but it does not seem to matter too much) received along with the announcement; but no doubt much more information will be published in the coming monthes.

Bernard Malgrange had a tremendous impact on the theory of complex differential equations, thus, indirectly, on the theory of q-difference equations. Quite a few essential results in this book were obtained trying to adapt work of Malgrange: this is of course

[11]As for what the book is on and what it isn't, see more details in the introduction.

[12]To name but a few: it seems to me that the mixing of *filtration by the slopes* and *graduation by the exponents* is not totally clarified by the use of the Grothendieck group in chapter 10; the phenomenon of confluence (chapter 11) could be proved more neatly using holomorphic functional calculus; and the determination of the fuchsian local Galois group in chapter 13 could somehow be illuminated by the theory of decomposition of tensor products of irreducible representations of linear groups; last, the possibility to attach a vector bundle to each q-difference system (chapter 7) is clearly under-exploited. All this is beyond my forces !

true of the q-analogues of Birkhoff-Malgrange-Sibuya theorems, but also actually of many things in chapters 7 to 10. More recently, his contribution to non linear differential Galois theory received a beginning of extension to q-differences in the work **[95]** by Anne Granier.

Malgrange never took a deep interest in q-difference equations, but, as a close friend of Ramis, he kept informed about them. He was the referee for my first articles and displayed a patience little deserved by them. His very professional attitude was a great help to me.

Quotation:
"Né le 6 juillet 1928 à Paris, ancien élève de l'Ecole Normale Supérieure, Bernard Malgrange a été Attaché de Recherche au CNRS, Maître de Conférences et Professeur à l'Université de Strasbourg et Professeur à la Faculté des Sciences d'Orsay avant de rejoindre l'Université de Grenoble en 1969, d'abord en tant que Professeur et ensuite comme Directeur de Recherches au CNRS. Membre de l'Académie des Sciences depuis 1977, récipiendaire de nombreux prix pour ses travaux profonds sur les équations aux dérivées partielles, la géométrie différentielle, les singularités de fonctions analytiques et la théorie algébrique des équations différentielles, il a participé très largement au rayonnement de l'Institut Fourier dans le monde mathématique. Le théorème de préparation différentiable restera pour toujours associé à son nom."

Introduction

0.1. General orientation of this book

The greatest single impetus in the theory of q-difference equations was probably Birkhoff's article [**22**]. After some sleepy (though not totally void) decades, the theory knew a revival at the very end of the previous millenium. This book revolves around one of the active threads of that revival during a quarter of a century.

0.1.1. Scope. There are many excellent books and survey articles about q-special functions (and also some about q-calculus and about difference algebra). This book is not one of them. It stands, in relation to the former, in more or less the same position as would a book on (linear, complex, analytic) differential equations with respect to a book on (classical[13]) special functions.

A theoretical approach is pursued. Functional equations are a powerful way to understand functions, and they in turn can be understood through underlying structures (more on this in 0.1.3). We shall try to detail and explain those structures as clearly and explicitly as possible, with many examples and exercises of application; however neither numerical nor symbolic algorithms will be displayed.

0.1.2. Origins and specificities of q-difference equation theory. While the Big Bang of q-difference equation theory is [**22**], the history of q-calculus goes back at least to Euler, maybe even to Fermat (see chapter 1). Apart from an untimely forerunner in Euler (the "fundamental trick", see 1.2), the first conscious use of q-difference equations seems to be due to Jackson [**118**] as a way to study Heine's basic hypergeometric series. Yet, according to Birkhoff's historical article [**23**] (see in particular p. 639), the Founding Fathers are Carmichael, Adams, Birkhoff himself, Trjitzinsky ...

Although in [**22**] Birkhoff puts "the three sister theories" of differential, difference and q-difference equations as far as possible on the same ground, he admits (p. 640 of [**23**]) that the latter is simpler. This is true and important and it can be understood at least in two ways:

- Any non trivial analytic automorphism of the Riemann sphere is a Mobius transformation, thus conjugate either to the translation operator $x \mapsto x+1$ (if it has a single fixed point) or to some dilatation operator $x \mapsto qx$, $q \neq 0, 1$ (if it has two fixed points). The former case is clearly a degeneration of the latter (the fixed points merge).
- Now look at the action of said automorphism on $\mathbf{S} \setminus \{\text{fixed point(s)}\}$. The quotient of $\mathbf{C}$ by the action $x \mapsto x+1$ is a cylinder, an open Riemann surface, while the quotient of $\mathbf{C}^*$ by the action $x \mapsto qx$ is a torus, a

[13]Throughout his book, I will generally use the word "classical" to qualify concepts and techniques related to ordinary differential equations, as opposed to q-difference equations.

compact Riemann surface. The latter is obviously an easier playground for complex analysis.

If we prefer a comparison with differential equations, we note that the relevant operator is an infinitesimal automorphism, the notion of fixed point is unclear; and the closest we get to a quotient is the "Malgrange circle of directions" S^1 (see 15.3.3 for explanation), not even a Riemann surface, thus not such a nice playground for complex analysis.

Compared to q-difference theory, plain difference theory is much more complicated and we shall have nothing to say about it (look at [**22, 72, 114, 236**] to get a feeling of how much more complicated it can be !). However, if we rather compare q-difference theory to differential equation theory, it offers two striking properties which make life easy:

- The domain of definition of local solutions can usually be expanded through q-dilatations and most equations admit *uniform* solutions over the whole of $\mathbf{C}^*$, which is of course not to be dreamed of in classical theory; see for instance 5.3.2.2, and, for a conceptual explanation, Praagman's theorem in 7.3.
- As for the *local* study, all analytic q-difference operators admit an *analytic* factorisation, while in the differential or difference case only a *formal* factorisation is possible. We call this fact Adams lemma, see 5.3.2, although it seems to have been independently rediscovered by Birkhoff and Guenther, [**24**], see chapter 15.

Yet, classical theory of (linear, complex, analytic) differential equations is a reliable guide (we frenchmen say "Fil d'Ariane") in that new maze, while trying to stay conscious at all times of similarities and disimilarities. A notable thread originating in classical theory is (under any of its thousand guises) Riemann-Hilbert correspondence (see chapter 12), which we shall rather call Riemann-Hilbert-Birkhoff correspondence since Birkhoff has renewed it so much in [**22**].

REMARK 0.1. In that respect, I must mention that the frequent reference to my book [**211**] just means that I presented there very classical results (none of them due to me !) but in a form that suits rather closely the points of view of the present book.

EXERCISE 0.2. One parallel is that differential equation theory is additive while q-difference equation theory is multiplicative. Try to find examples of that in the text. (For instance: definition of basic functions, of exponents, properties of slopes, Fuchs relation . . .).

0.1.3. Technical bias.

> Thus, after due respect has been paid to the algebraic verifications in their various shapes, I am still convinced again all Puritan doctrines that the analytic method is the least artificial, affording the deepest insight and best in keeping with our program: to solve concrete problems by means of general ideas which shed light upon a much wider range of mathematical facts that were needed for our immediate purpose.
>
> Hermann Weyl, "The Classical Groups", chap. VII, §7.

As can be guessed from the previous considerations (and the above quotation), we favour a wholly transcendental approach. The course will involve only *complex* *analytic linear q-difference equations*[14] and for them it seems that the strongest results have been obtained through function theoretic methods, appropriately seasoned with an adequate proportion of algebra.

A fundamental restriction is that $|q| \neq 1$. The case $|q| = 1$ is much more complicated and not so much developed (see however for instance **[58]** for a significant advance and, for example, **[225]** for the usefulness of that case). We avoid the arithmetic theory **[56, 57]** and the more algebraic approach of **[236]** based on Picard-Vessiot theory (this, however, should be fixed later - see herebelow 0.1.4). To return to our mentioned restriction, and to make it more precise, we shall require:

$$0 < |q| < 1.$$

The case $|q| > 1$ boils down to the previous one through easy procedures which will be commented later in the course of the text.

Although the founding fathers used multivalued functions, we shall take advantage of the possibility explained above in 0.1.2 to totally avoid them and work with uniform functions. This idea originates in the programmatic text **[181]** by Ramis (itself related to a suggestion of Birkhoff at the end of **[22]**). (More will be said on that choice in 12.1.1.)

When tackling global problems, the base field will be $\mathbf{C}(x)$. When tackling local matters, it will be either $\mathbf{C}(\{x\})$ or $\mathbf{C}((x))$. It is possible to work over rings like $\mathbf{C}\{x\}$, $\mathbf{C}[[x]]$ or $\mathbf{C}[x, x^{-1}]$ but this leads to unnecessary algebraic complications (torsion modules, etc) in our setting - although it is plainly appropriate[15] for instance when going beyond one variable as for instance Sabbah does in **[203]**. However, it is true that our choice to avoid q-difference rings or algebras sometimes leads to overelaborate explanations.

We have founded the algebraic part of the theory on q-analogues of differential modules, and, to a lesser extent, of $\mathcal{D}$-modules (see chapter 8). We did not find it necessary to introduce "discrete connections" of the kind used for instance by Tarasov and Varchenko in **[230]**.

0.1.4. A projected "Companion volume". Given all the important topics that we fail to adress, it was decided (along with a few colleagues working in the domain) that we would prepare a kind of "Companion volume" (in the sequel, to be referred as [CV]) at a higher level of sophistication and tackling the following matters:

(1) Asymptotics and summation theory for irregular equations (here barely touched in chapters 14 and 15).
(2) Important q-special functions, notably:
   - q-hypergeometric functions,
   - Rogers-Ramanujan continued fractions.
   - orthogonal polynomials,

   Note that the first two items do appear in the book at various places as illustration (and we initiate a study of the first in chapter 4).

[14] There is a tradition originating, it seems, in physics, which considers difference and q-difference equations as some kinds of recurrence relations: see *e.g.* **[175]** or, to some extent, **[89]** and **[124]**. Here, we stick to the functional interpretation and enjoy the full power of function theory.

[15] A serious treatment of holonomy also requires base rings or algebras, so our baby study in 3.2.4 is really a low level one.

(3) q-Painlevé equations, from the analytic and from the geometric point of view.
(4) Local classification and Galois theory for irregular equations with arbitrary slopes (in chapters 14 and 15, we solve the corresponding problems for equations with integral slopes only; the general case is more complicated).
(5) Comparison of the present function theoretic approach to Galois theory with the more algebraic Picard-Vessiot approach.
(6) Effective methods, procedures and algorithms in resolution, classification and Galois theory.
(7) Simultaneous solutions of q-difference equations and of differential equations, hypertranscendence, simultaneous solutions of q-difference equations for more than one q.

0.2. Contents

Broadly speaking, the book has three parts. Chapters 1 to 4 are meant as an initiation to the q-difference world. Chapters 5 to 10 are the technical heart, they display the tools of the trade. Chapters 11 to 15 aim at substantial applications. (Of course, the distinction is not so clear-cut: those last five chapter also display important tools and chapter 4 already tackles significant applications.) There are appendices (see 0.3.1.1). At the end of the introduction (from 0.3 on) we provide a practical guide to the book, including organisation, prerequisites, notations, conventions, etc.

0.2.1. Initiation. Chapter 1 contains a series of historical examples predating the real period of q-difference equations; so, in some sense, they are prehistorical. They are mainly intended to get the reader accustomed to that Brave New World and its rich flavours; but they do come with explicit calculations and arguments.

Chapter 2 introduces the "basic bricks", *i.e.* the most elementary q-special functions needed to solve all q-difference equations. They rest on classical special functions, for which reminders are provided in appendices A and B. We also begin there an initiation to the role of sheaves and vector bundles in q-difference equation theory (see why further below in 0.2.4).

Chapter 3 introduces elementary calculation techniques from difference algebra and from q-calculus.

Chapter 4 addresses equations of order 1 (in this way, it is but an extension of the previous chapter) and equations of order 2; and there, it already involves significant applications, like basic hypergeometric series. In order to emphasize the specificities of the latter, appendix C summarizes the basics on classical hypergeometric functions. We go on with the initiation to the role of sheaves and vector bundles (the real thing will start in 7.3 and in 9.3).

0.2.2. Tools of the trade. Chapter 5 adapts the classical methods inherited from the founding fathers to our framework based on uniform functions (in this way, it rests on chapter 2). It also takes advantage of algebraic methods from differential and difference algebra (noncommutative polynomials) to expound our first important theorem, Adams lemma; and to introduce in his first avatar an essential tool, the Newton polygon.

Chapter 6 originates in work of Bézivin and of Ramis. It is more "modern" than chapter 5 in that it replaces resolution by the measure of obstructions to the possibility of resolution. It uses some functional analysis and very elementary homological methods, all recalled in appendix D.

Chapter 7 displays the classical method of vectorialisation of scalar equations, well known in the study of differential equations. It is probably due (once again !) to Birkhoff. The complete understanding of relations between scalar equations and systems also requires some elementary homological methods, summarized in appendix E. It is also the right place to start to systematize the use of sheaves of solutions and the attached vector bundles (which have, actually, already been touched upon in special cases).

Chapters 8, 9 and 10 are, in my opinion, the technical heart of our toolbox. First we start the heavy use of linear methods in chapter 8 (modules over noncommutative rings), and this requires some linear and multilinear algebra, see appendix F. We complete it with a more detailed algebraic study in chapter 9. Then we apply it to elucidate the formal and analytic structure of (complex, analytic, linear) q-difference equations in chapter 10.

0.2.3. Applications to "the real world". As in the classical case, there is a first division of the world between *regular singularities* (when solutions have moderate growth; they come here under the heading of *fuchsian equations and systems*) and *irregular singularities.*

In our framework, the only possible local studies are at 0 and at ∞ (which boils down to the previous one); this is explained in chapters 11 (local study of fuchsian systems) and 12 (global study), where the original contribution of Birkhoff [**22**] is modernized, systematized and extended. This is then completed by the Galois theory in chapter 13, where we begin to require some more sophisticated tools (tannakian duality) recalled in appendix G. Chapter 12 also involves an opening towards a non linear theme, that of q-Painlevé equations, which will be more seriously adressed in [CV].

In chapter 14 we treat irregular *scalar equations.* The scalar form is appropriate for the finer analytic theory, but we unhappily can only touch upon it here: this will be hopefully compensated by [CV]. Like the following one, this chapter uses some cohomology of sheaves, which is summarized in appendices H (abelian cohomology) and I (non abelian cohomology).

Chapter 15 adresses irregular *systems*, which are an appropriate framework for classification and Galois theory, so we require again information from appendix G; but classification also requires sheaf-cohomological methods from appendices H and I.

0.2.4. Why the fuss about vector bundles and sheaves. Theorems on factorisation of holomorphic matrices (by Hilbert, Plemelj, Birkhoff, Cartan ...) have been for a long time associated to the study of functional equations. Röhrl [**194**] formulated and solved a Riemann-Hilbert problem along those lines in terms of vector bundles (also see [**86**, chap 3 §31], [**173**], [**236**, 12.3.1], [**211**, chap 12, theorem 12.8]). So the use of holomorphic vector bundles here seems unavoidable. On the other hand, it seems to me that their elementary study is easier in terms of the associated locally free sheaves, since the formalism of sheaves offers so much flexibility. Whence the sections 7.3 and in 9.3 and the appendices H and I.

0.2.5. The appendices. Their content has been described above. Their role is explained in 0.3.1.1.

0.3. Some practical tips

0.3.1. Organisation and *mode d'emploi*.

0.3.1.1. *Prerequisites.* The main prerequisites are general algebra and analysis as they are taught in the first three years of University (or College if you prefer). However, specific higher level math is needed at some places, most of the time at the level of fourth year. I expect that each particular reader has been confronted to some of them, but not all.

Therefore, each time it is so, I have tried to summarize it in the form of an appendix, like I usually do when delivering the corresponding graduate course. The appendix is meant to *avoid* the necessity to dwell on the subject, by providing a user-friendly access to terminology and results.

0.3.1.2. *Exercises.* Two kinds of exercises are presented: some serve as an illustration (*e.g.* examples, counterexamples, explicit calculations, etc). Some may propose deepening or extensions of the main text. Many solutions or hints will be gradually posted on the AMS webpage related to this book during the year following its publication.

0.3.1.3. *Indexes.* I have prepared a terminological index, a notation index and an index of names. They take in account the main text only, not the appendices. The terminological index only mentions terms special to the book, terms of general mathematical use are listed further below in 0.4.2. The index of names mentions names only for relevant contributions, for instance expressions like "Riemann surface", "Galois groups" and "Hilbert theorem 90" do not give rise to items like "Galois", "Riemann" or "Hilbert". I found no definite rule for the notation index, which is maybe a mess (containing at times entire formulas ...): I only hope that a desperate reader may find at times more convenient to browse through 12 pages of symbols than to browse through more than 650 pages of the book.

0.3.1.4. *Errata.* I cannot hope to have corrected all the typographical and more substantial errors that appeared in the long process of making this book. A list of errata will be maintained on the webpage mentioned above.

0.4. General notations and conventions

0.4.1. Some general conventions. The end of a proof, of its absence, is marked by the symbol $\Box$.

Notation $A := B$ means that the term A is defined by formula B.

New terminology is written in *emphatic style* when first defined. Example:

The q-difference module M is said to be *fuchsian (at* 0*)* if it is pure isoclinic of slope 0, *i.e.* if $S(M) = \{0\}$.

Note that a definition can appear in the course of a theorem, an example, an exercise, etc.

We use commutative diagrams. For instance, the commutativity of the following diagram means that $\phi_V \circ \rho_V^U = {\rho'}_V^U \circ \phi_U$:

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\phi_U"] \arrow[d, "\rho_V^U"'] & \mathscr{F}'(U) \arrow[d, "{\rho'}_V^U"] \\ \mathscr{F}(V) \arrow[r, "\phi_V"'] & \mathscr{F}'(V) \end{tikzcd}$$

0.4.2. Some general notations.

0.4.2.1. *Sets of numbers.* • $\mathbf{C}, \mathbf{R}, \mathbf{Q}, \mathbf{Z}, \mathbf{N}$: complex, real, rational numbers; then rational, natural integers.
• $\mathbf{C}^*, \mathbf{R}_+, \mathbf{R}_+^*$: non zero, non negative, strictly positive.
• $\mathbf{U}, \mu_n, \mu_n^*, \mu_\infty$: modulus one, n^{th} root of unity, primitive n^{th} root, arbitrary root.
• $\Re, \Im$: real and imaginary part. (The image of a map is denoted Im f.)

0.4.2.2. *Spaces.* • $\mathbf{S}, \mathbf{C}_\infty$: Riemann sphere, the same deprived of 0.
• $\mathcal{H}$: Poincaré half-plane (defined by $\Im z > 0$).
• $\overline{\mathrm{D}}(a,R), \overset{\circ}{\mathrm{D}}(a,R), \overset{\bullet}{\mathrm{D}}(a,R), \partial\mathrm{D}(a,R)$: closed, open, punctured disk, circle.
• $\overset{\circ}{\mathcal{C}}(r,R), \overline{\mathcal{C}}(r,R), \mathcal{C}(r,R)$ open, closed, semi open annulus $r < |z| \leq R$.
• $\Lambda, \Lambda_\tau, \mathbf{E}_\Lambda$: a lattice of $\mathbf{C}$, the lattice $\mathbf{Z} + \mathbf{Z}\tau$, the quotient $\mathbf{C}/\Lambda$.
• $\mathrm{div}, \mathrm{div}_V$: divisor of a function, of its restriction to V.
• $\deg, \mathrm{ev}$: degree of a divisor, evaluation (if defined on an abelian group).

0.4.2.3. *Sets of functions.* • $\mathscr{O}(U), \mathscr{O}(U,a), \mathscr{M}(U), \mathscr{M}(U,a)$ holomorphic functions and germs, meromorphic functions and germs.
• $\mathbf{C}[[x]], \mathbf{C}\{x\}, \mathbf{C}((x)), \mathbf{C}(\{x\})$ rings of formal and analytic series, their quotient fields.
• $\mathbf{C}[x], \mathbf{C}(x), \mathbf{C}[x]_d$ polynomials, rational fractions, polynomials of degree $\leq d$.
• $v_a(f), \mathrm{Res}_{x=a}$ valuation or order of f at a, residue.
• f is *regular* at a if $v_a(f) = 0$.
• $\|f\|_K$: maximum of $|f(x)|$ on compact K.
• $D := d/dx, \delta := xd/dx$: plain and Euler differential operator.
• $T_n f, \overline{T}_n f$: truncatures of series, so that $\overline{T}_n f - T_n f$ is the x^n term.

0.4.2.4. *Matrices, linear algebra.* • $\mathscr{L}_K(V,W), \mathscr{L}_K(V), \mathrm{GL}(V)$: K-linear maps, endomorphisms, automorphisms.
• $\mathrm{Mat}_{m,n}(K), \mathrm{Mat}_n(K), \mathrm{GL}_n(K)$: corresponding sets of matrices.
• $\mathrm{D}_n^*(K), \mathrm{Diag}(c_1, \ldots, c_n)$: invertible diagonal matrices (set, individual).
• $\mathrm{Sp}(A), [A,B]$: spectrum[16], commutator $AB - BA$.
• Sing $M(x)$: singular locus, *i.e.* where M is undefined or not invertible.
• $\begin{bmatrix} C_1 & \cdots & C_n \end{bmatrix}, \begin{bmatrix} L_1 \\ \vdots \\ L_m \end{bmatrix}$: denotations of a $m \times n$ matrix by columns, by lines.
• $|||A|||$: matricial norm associated to a norm $\|X\|$ on $\mathbf{C}^n$.

[16]Sometimes, the spectrum is seen as a plain set and for instance, writing 0_n the null $n \times n$ matrix, Sp $0_n = \{0\}$. Sometimes, it is seen as a *multiset* (elements have multiplicities) and then Sp $0_n = \{\{0, \ldots, 0\}\}$ (counted n times).

0.4.2.5. *Categories,sheaves and vector bundles.* • $\mathrm{Ob}(\mathscr{C}), \mathrm{Hom}_{\mathscr{C}}(X,Y), \mathrm{Id}_X$: class of objects, sets of morphisms, identity morphisms in a category.
• $\rightsquigarrow$: funny arrow reserved for functors acting on objects.
• $\mathcal{V}ect^f_{\mathbf{C}}, \mathcal{R}ep_{\mathbf{C}}(G), \mathcal{R}ep^f_{\mathbf{C}}(G), \mathcal{R}ep_{rat}(G)$: category of finite dimensional complex vector spaces, of complex representations, of finite dimensional complex representations, of rational representations.
• $\varprojlim, \varinjlim$: inverse and direct limit.
• $\mathscr{F}, \mathfrak{F}$: locally free sheaf and associated bundle.
• $\rho^U_V, \mathscr{F}_a, \mathfrak{F}_a$: restriction map, stalk (or fiber) for a sheaf, for a vector bundle.

0.4.2.6. *Miscellani.* • $n >> -\infty, n >> 0, n << a, \ldots$: for n bounded below, for n big enough, etc.
• $\{\{c_1, \ldots, c_n\}\}$: multiset, or bag (of roots, of eigenvalues ...).
• Empty sum is 0; empty product is 1 or I_n, etc; $\min \emptyset, \max \emptyset$ respectively biggest or smallest possible value.
• $\simeq, \sim, \mathrm{cl}, \approx$: isomorphism, equivalence relation, equivalence class, *ad hoc* relation of "similarity".
• $[\Gamma,\Gamma], \Gamma^{ab}, \mathrm{Hom}_{gr}, \mathrm{Hom}_{grtop}, \Gamma^{\vee}$: commutator subgroup, abelianized, set of morphisms of groups, of continuous morphisms, proalgebraic group of morphisms to $\mathbf{C}^*$.
• K^σ, R^σ: fixed subfield, subring under an automorphism.
• E^G fixed subset of E acted upon by the group G.

0.4.2.7. *The q-world.* • $q \in \mathbf{C}^*$ s.t. $|q| < 1$ but for some explicit exceptions.
• $\equiv$: most of the time congruence modulo $q^{\mathbf{Z}}$ in $\mathbf{C}^*$.
• $\tau \in \mathcal{H}$ s.t. $q = e^{2\mathrm{i}\pi\tau}$, $q^\alpha := e^{2\mathrm{i}\pi\tau\alpha}$, $q^{\mathbf{R}} := e^{2\mathrm{i}\pi\tau\mathbf{R}}$.
• $\sigma_q f(x) := f(qx)$, $D_q := \dfrac{\sigma_q - 1}{(q-1)x}$, $\delta_q := \dfrac{\sigma_q - 1}{q-1}$.
• $\mathscr{L}_{\sigma_q}(V,W)$: σ_q-linear maps, *i.e.* group morphisms s.t. $f(ax) = \sigma_q(a)f(x)$.
• $\mathbf{E}_q := \mathbf{C}^*/q^{\mathbf{Z}}$: group and Riemann surface, $\pi : \mathbf{C}^* \to \mathbf{E}_q, c \mapsto \overline{c}$.
• $\mathcal{C}_q := \mathcal{C}(|q|, 1)$: fundamental annulus.
• $c = q^\epsilon(c)\overline{c}$, $\epsilon(c) \in \mathbf{Z}$, $\overline{c} \in \mathcal{C}_q$.
• $[a;q] := aq^{\mathbf{Z}}$, $[a;q]_0 := aq^{\mathbf{N}}$, $[a;q]_\infty := aq^{-\mathbf{N}}$: discrete q-spirals and half q-spirals.
• $(a;q)_n, (a;q)_\infty, [\alpha]_q, [\alpha]!_q, \begin{bmatrix} n \\ l \end{bmatrix}_q$: q-Pochhammer symbols (in two guises), q-numbers, q-factorials, q-binomial coefficients (section 3.3).
• $\theta_q, e_{q,c}, \ell_q$: q-theta function, q-character,q-logarithm (chapter 2).
• $A \underset{K}{\sim} F[A] := (\sigma_q F)AF^{-1}$, where $F, A \in \mathrm{GL}_n(K)$: gauge tranformation and equivalence over K.